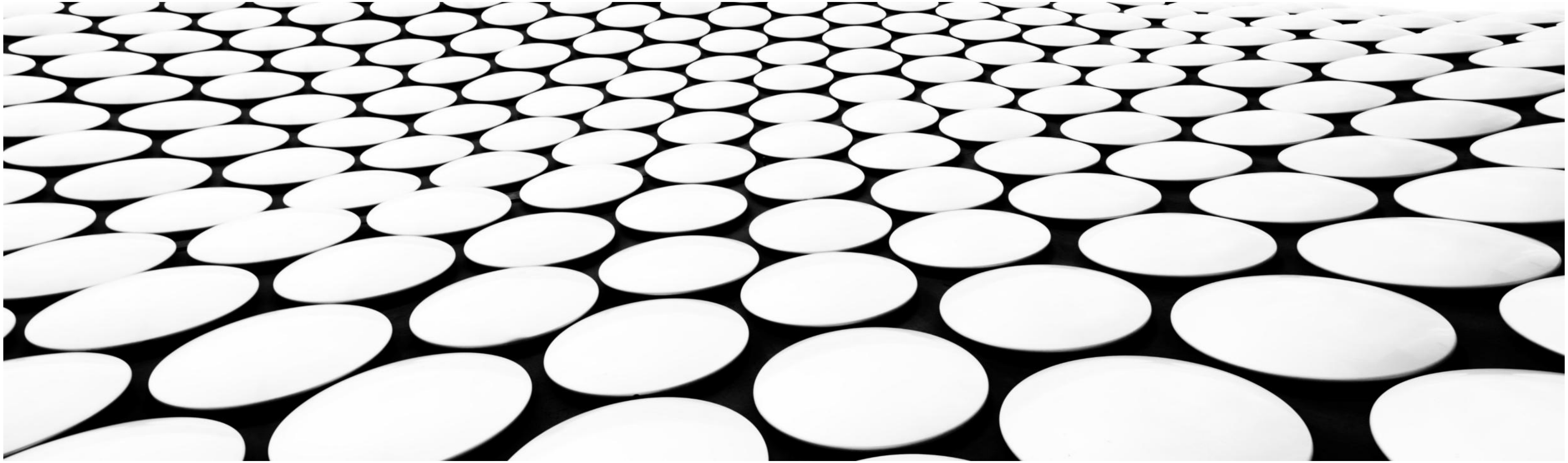
MATH CIRCLE AT FAU

12/14/2024



THE ISLAND OF KNIGHTS AND KNAVES

Here we are on the island of knights and knaves; The knights who can only tell the truth, the knaves who always lie.

You visit the island and meet Slippy, a **knave**.

Slippy tells you: *All my hats are green.*

From what he said, knowing he is a lying knave, what can we conclude **for sure** from Slippy's statement?

A. Slippy has at least one hat.
B. Slippy has only one green hat.
C. Slippy has no hats.
D. Slippy has at least one green hat
E. Slippy has no green hat.

Alex Belos, in The Guardian

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The answer is A

Here is a little logical quirk that gave a lot of trouble to medieval scholars. The negation of

All my hats are green

is

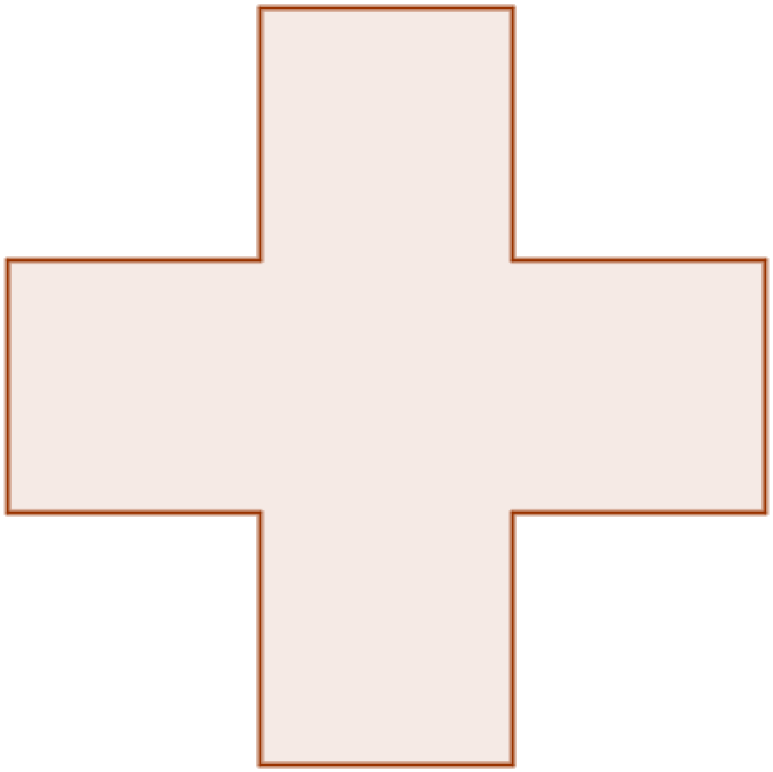
Some of my hats are not green.

Does this statement hold **for sure** in cases B, D, E? The answer is no.

The case C is more delicate. The funny thing is that if Slippy has no hats, then all his hats are green (or red, or whatever you want them to be) So if C holds, then Slippy is telling the truth. C must be false.

THE CRUX OF THE MATTER

The cross pictured on the right, having all arms of the same length, can be divided by two straight cuts into four (or five?) pieces that can be assembled to form a square. Try to find the cuts.



THE CRUX OF THE MATTER - SOLUTION

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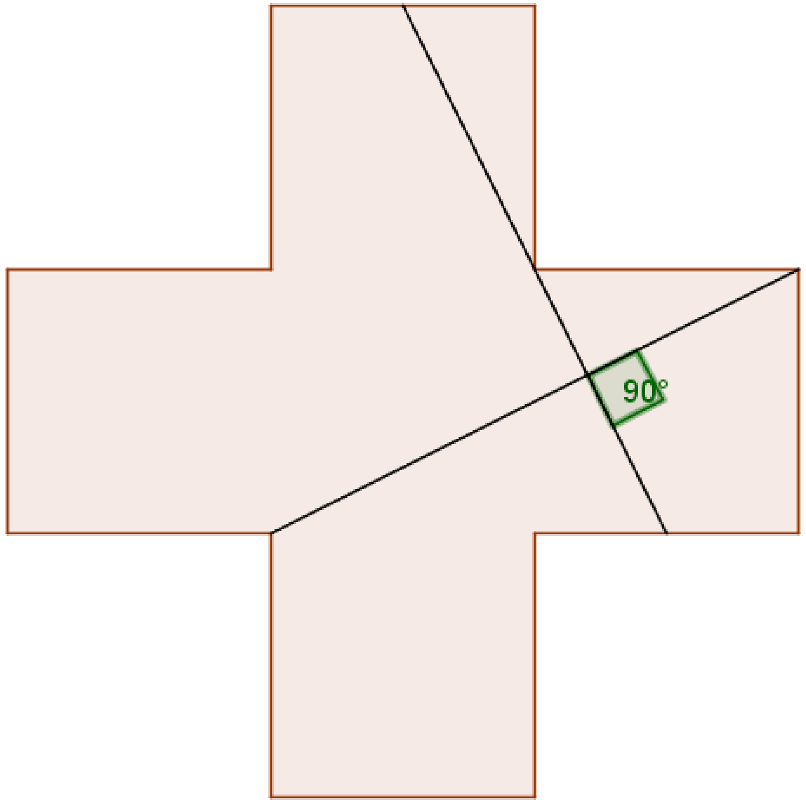
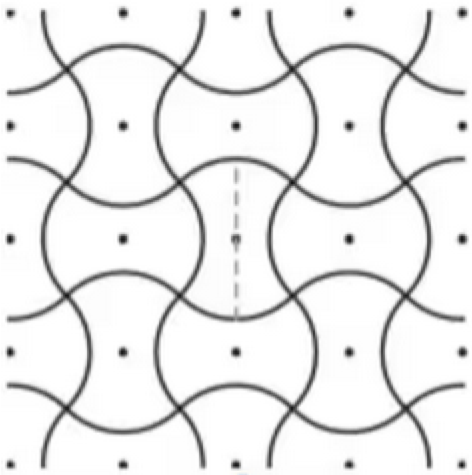


TWISTED TILES



The edges of these identical tiles are quarter circles, and their centres are the points marked. Determine the area of a tile, measured in cm^2, given that the height of a standing tile is 12cm.

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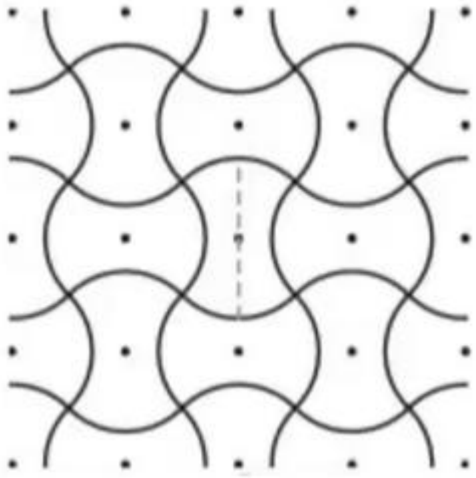

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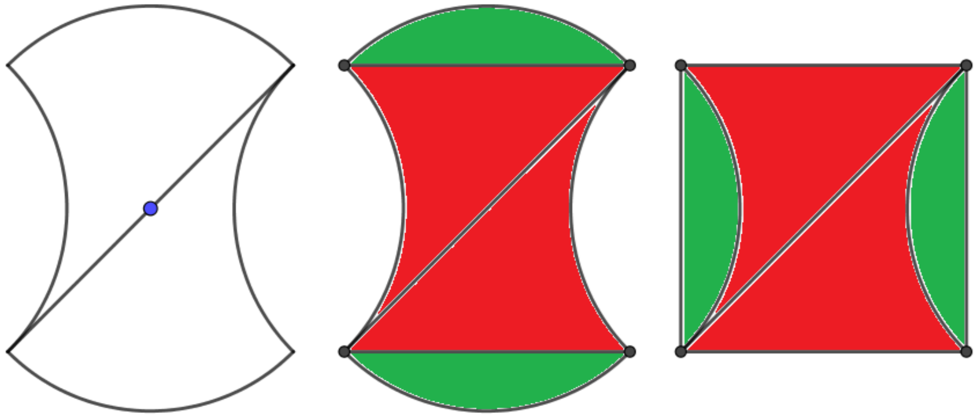

The radius of the quarter circle is $\frac{12}{2} = 6$. Thus the segment of the first picture is 12 cm. The other pictures show that the tile has the same area as a square of diagonal 12 cm. Applying Pythagoras one sees the side of the square is $\ell = \frac{12}{\sqrt{2}}$ so that the area is $\boldsymbol{\ell^2 = 72cm^2}$.

BERTIE WOOSTER’S WANDERINGS

Every weekday Mr. Wooster returns from his club in the city (where he spends the day drinking tea and gossiping) by train, to the train station of the town where he lives. His butler Jeeves is supposed to pick him up by car and drive him home. No time is to be wasted; Jeeves is to leave the car, so he exactly arrives at the train station the moment the train pulls in. Mr. Wooster jumps into the car, and they drive immediately home. One day the train arrives early, Jeeves isn't there yet, and Mr. Wooster decides to walk home. After walking for half an hour he meets Jeeves on the way to pick him up. He gets into the car, and they arrive at the home 20 minutes earlier than usual. How many minutes early was the train?

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Since Jeeves drove 20 minutes less than usual, he drove 10 minutes less in each direction. As he meets Bertie, the train has been gone for 30 minutes. Normally Jeeves would drive another 10 minutes to get to the station. By then the train would have been gone for $30 + 10 = 40$ minutes.

Answer: 40 minutes.

THE DAY OF THE WALKERS

- Suppose there are exactly 9 towns in a very small country and **all distances between the towns are different**. So, for example, if two towns are at a distance of 1 mile from each other, all other towns are at a distance other than 1 mile from these two, and no other pair of towns are at a distance of 1 mile from each other.
- One morning, a person starts in each town and walks towards the **nearest** town.
- Prove:

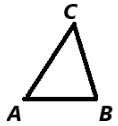
(a) There are two towns A and B such that a person from A walks to B, and a person from B walks to A.

(b) There is a town that nobody walks to.

HINT: Try it first with 3 towns, 5 towns, …

THE DAY OF THE WALKERS - SOLUTION

- We have 9 towns and 9 walkers. Of these towns two will be closest to each other (recall no two are at the same distance), call them A and B. The walker of A will go to B, the walker of B to A. We are left with 7 walkers and 7 towns. This takes care of (a).
- Suppose one of the seven remaining walkers goes to A or B. Then we are left with 6 walkers and still with 7 towns; one will not be visited.
- But if none of the 7 remaining walkers goes to A or B, we now have the same problem we had with 9 towns reduced to a problem with 7 towns.
- Of these 7 towns there will be two closest, call them C, D. The walker from C goes to D, from D to C. Etc.
- The problem reduces to the case of 5 towns, finally to 3 towns. The picture for 3 towns looks like:



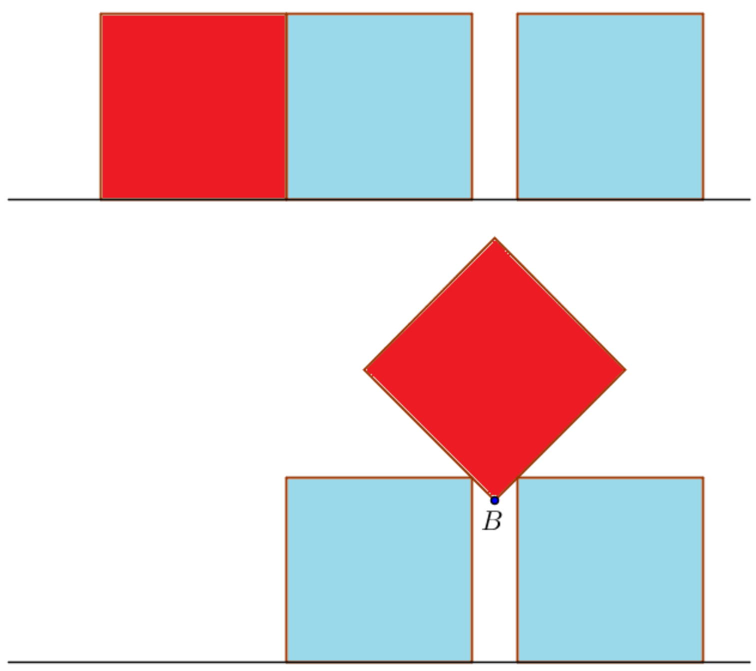
- The walker from A goes to B, the one from B to A; the walker from C goes to A or B; nobody walks to C.
- This is a nice illustration of the concept of mathematical induction.

QUIRKY SQUARES

Three 4-inch squares are placed with their bases on a line; the first square and the second square are side by side, the third one is one inch to the right of the second one, as in the top picture..

The leftmost square is then lifted out and rotated $45°$, then it is centered and lowered into the space between the second and third square until it touches both of these squares, as in the bottom picture.

How many inches is the point B from the line on which the bases of the squares were originally placed?

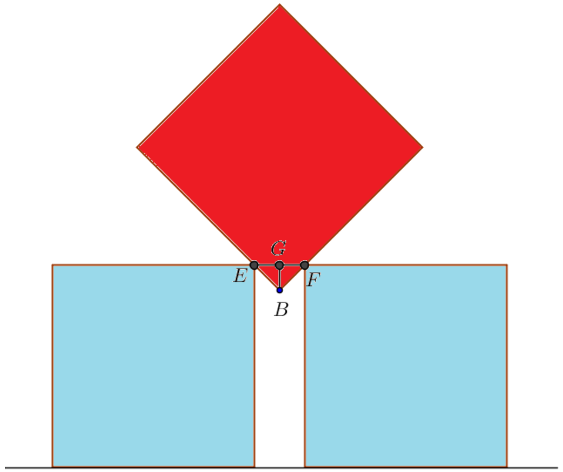


QUIRKY SQUARES

Three 4-inch squares are placed with their bases on a line; the first square and the second square are side by side, the third one is one inch to the right of the second one, as in the top picture..

The leftmost square is then lifted out and rotated 45°, then it is centered and lowered into the space between the second and third square until it touches both of these squares, as in the bottom picture.

How many inches is the point B from the line on which the bases of the squares were originally placed?



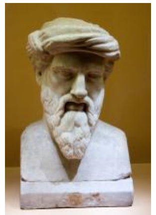
Consider the bottom picture with a few more points marked. By symmetry, and a bit of elementary geometry, one sees that $\Delta\, EFB$ is an isosceles right triangle of base 1 inch. Its altitude GB is then seen to be ½ an inch long. Thus, B is ½ inch lower than the top of the squares. It follows that the distance of B from the base line is

$$\mathbf{4 - 0.5 = 3.5}\ \textbf{inches}.$$

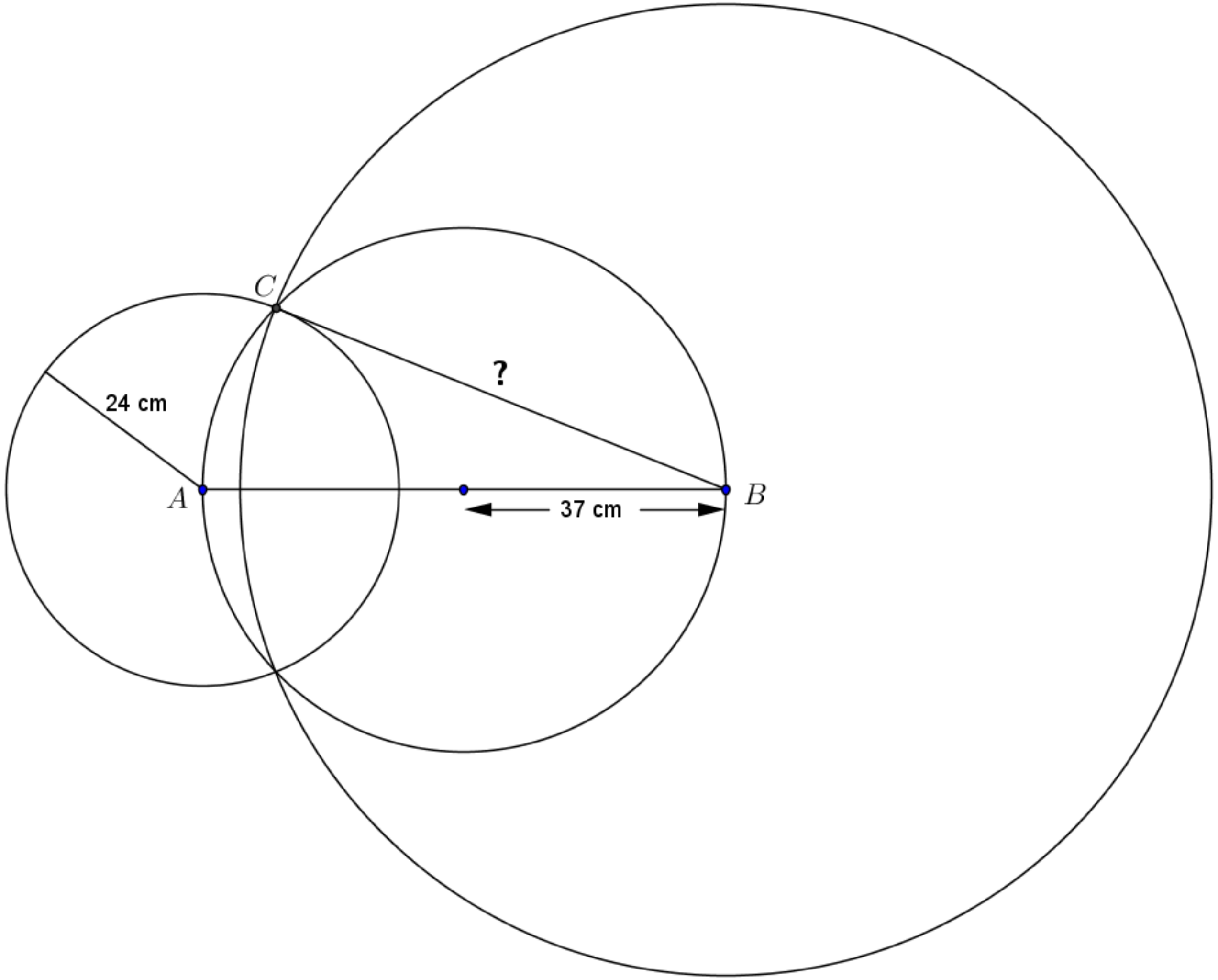
LOOKS HARD, BUT IS IT?

The circle of diameter AB has a radius 37 cm long. A circle of radius 24 cm. is drawn centered at A and it intersects at C a circle centered at B.

What is the radius of the circle of center B?.



A bust supposed to be (but almost certainly isn't) of Pythagoras. Here for inspiration.



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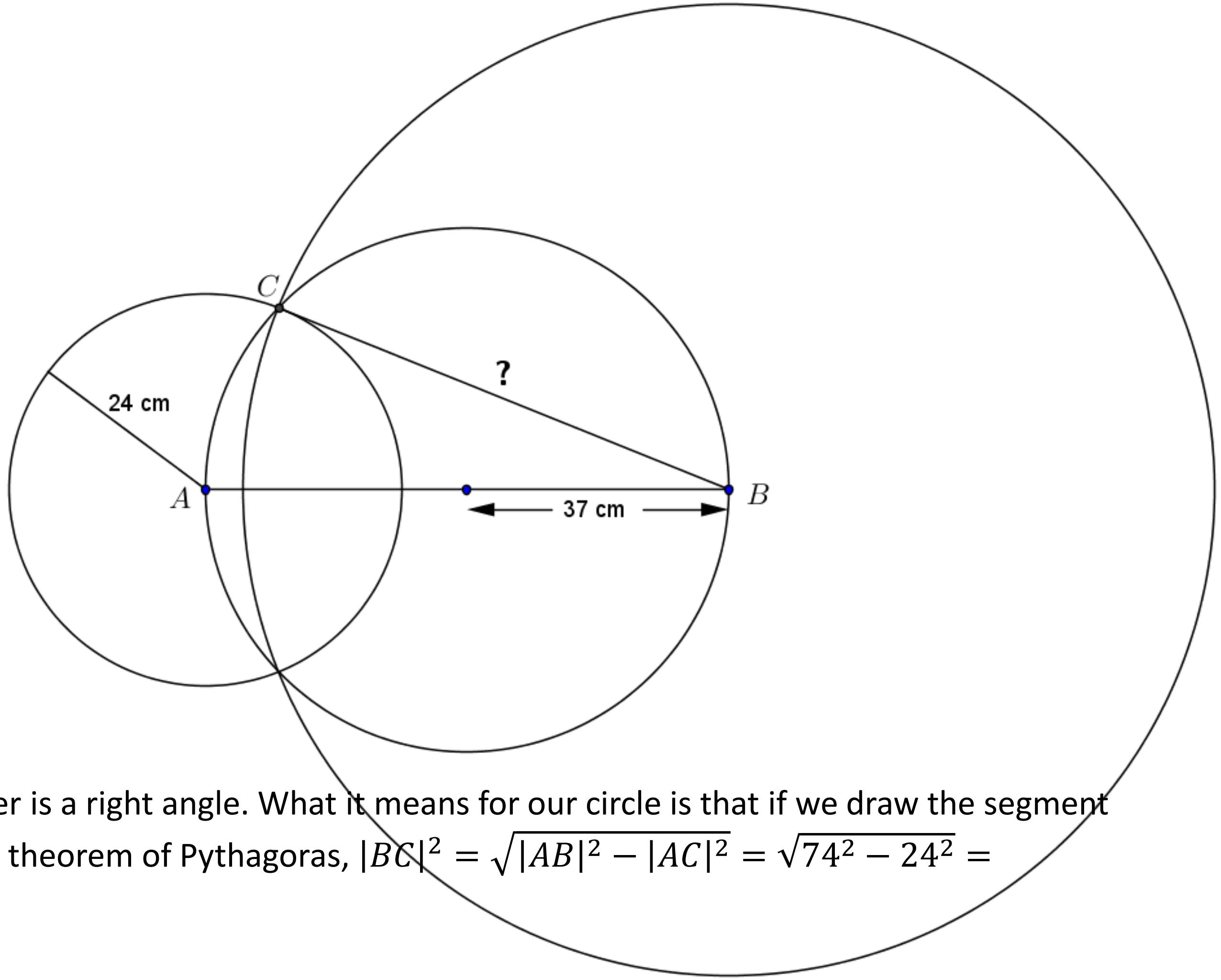
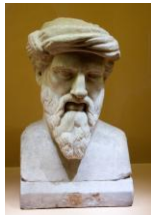




Solution: In any circle, the angle subtending a diameter is a right angle. What it means for our circle is that if we draw the segment from A to C, triangle ABC has a right angle at C. By the theorem of Pythagoras, $|BC|^2 = \sqrt{|AB|^2 - |AC|^2} = \sqrt{74^2 - 24^2} = \sqrt{4900} = 70$.

The radius is **70 cm** long.

ROOTING IT OUT

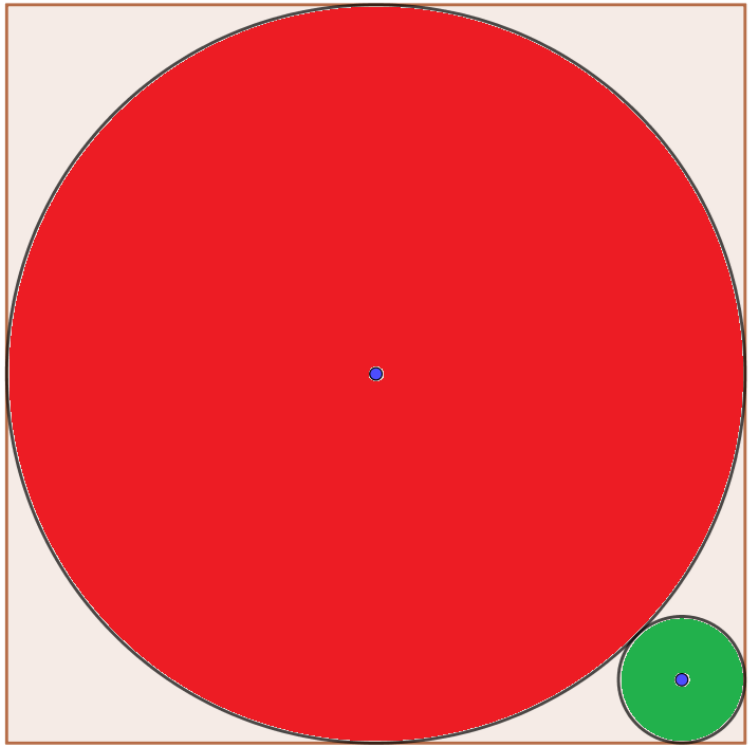
A red circle is inscribed in a square of side length 6. The green circle is tangent to the red one and to two sides of the square.

What is the radius of the green circle?

Possible answers are:

A. ½
B. $9 - 6\sqrt{2}$
C. $18\sqrt{2} - 25$
D. $6 - 4\sqrt{2}$
E. $6\sqrt{2} - 8$

You must justify your answer!



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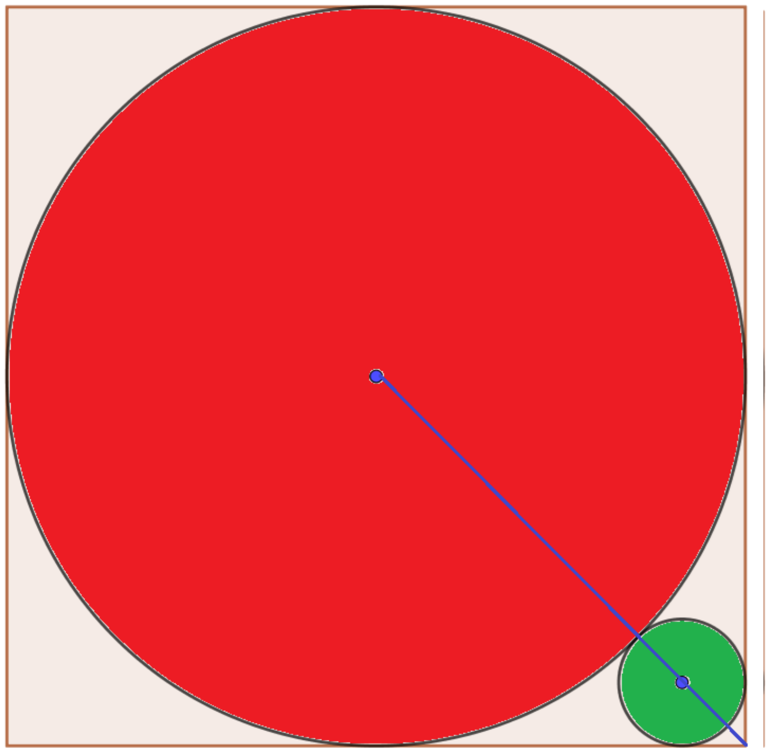

SOLUTION: Let r be the radius of the small circle, R the radius of the large circle. Clearly $R = \frac{6}{2} = 3$. We compute the length of the half diagonal of the square in two ways (the blue segment in the picture). By Pythagoras, the full diagonal is $6\sqrt{2}$ long, so the semi-diagonal is $3\sqrt{2}$ long. Its length is also $R + r + d$, where d is the distance from the center of the small circle to the lower rightmost vertex of the square. Pythagoras again tells us that $d = r\sqrt{2}$. Equating the two values for the length of the semi-diagonal,
$3\sqrt{2} = 3 + \left(1 + \sqrt{2}\right)r$. We can solve to get

$$r = \frac{3(\sqrt{2}-1)}{\sqrt{2}+1} = \frac{3\left(\sqrt{2}-1\right)^2}{(\sqrt{2}+1)(\sqrt{2}-1)} = 9 - 6\sqrt{2}.$$

Rationalizing we get that B is the correct answer.