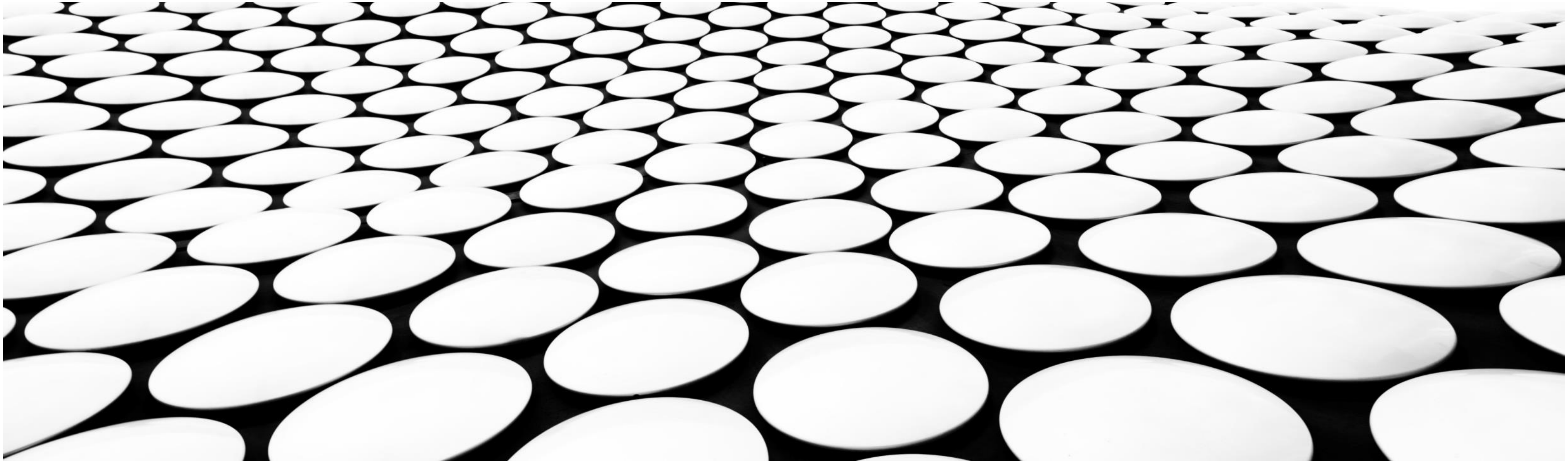
MATH CIRCLE AT FAU

11/16/2024



THE ISLAND OF KNIGHTS AND KNAVES

Here we are on the island of knights and knaves; The knights who can only tell the truth, the knaves who always lie.

You visit the island and meet one of the locals, Al.

Al tells you: “I love dogs.”
He then goes on to tell you
“If I love dogs then I love cats

Is Al a knight or a knave?

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Is Al a knight or a knave?

Solution: A statement “If p then q,” or the equivalent one “p implies q” is false if and only if p is true and q is false.
This embodies a fundamental article of mathematical faith: Truth cannot imply false.

Al is a knight

THE ISLAND OF KNIGHTS AND KNAVES

We are still on the island of knights and knaves; The knights who can only tell the truth, the knaves who always lie.

You visit the island and meet three locals, Ali, Baba, and Chippy..

Ali tells you: “Baba is a knight.”

Baba tells you: “If Ali is a knight, so is Chippy.”

What are Ali, Baba, and Chippy?

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Ali tells you: “Baba is a knight.”

Baba tells you: “If Ali is a knight, so is Chippy.”

What are Ali, Baba, and Chippy?

Solution: If Ali is a knave, then Baba is a knave. But then the premise of what Baba says is false, so, paradoxically?, the statement is true. CONTRADICTION with Baba being a knave.

Ali, Baba, and Chippy are knights.

LATIN TABLEAU

In a Latin tableau, each row must contain some permutation of he numbers from 1 to r, where r is the length of that particular row. Each column must contain some permutation of the numbers from 1 to c, c being the height of that particular column. On the right there is a Latin tableau, except that some entries have been erased. Your job is to restore the missing entries.

			6				5		
	7								2
		9				1			
8				3	6				
4									
				5					
	3								
		1							

(Puzzle by Timothy Y. Chow)

LATIN TABLEAU

10	9	8	6	7	2	4	5	3	1
9	7	10	8	6	4	5	3	1	2
6	10	9	7	8	5	1	4	2	3
7	8	6	5	4	3	1	2		
8	5	7	4	3	6	2	1		
5	6	4	2	1	3				
4	1	5	3	2					
2	4	3	1	5					
1	3	2							
3	2	1							

HERE IS ANOTHER ONE, TO DO AT HOME.

						5			2
								1	
10					6				
		5	8			2			
6		1							
	6				1				
					4				
		3							
1									
			3						

(Puzzle by Timothy Y. Chow)

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SOLUTION

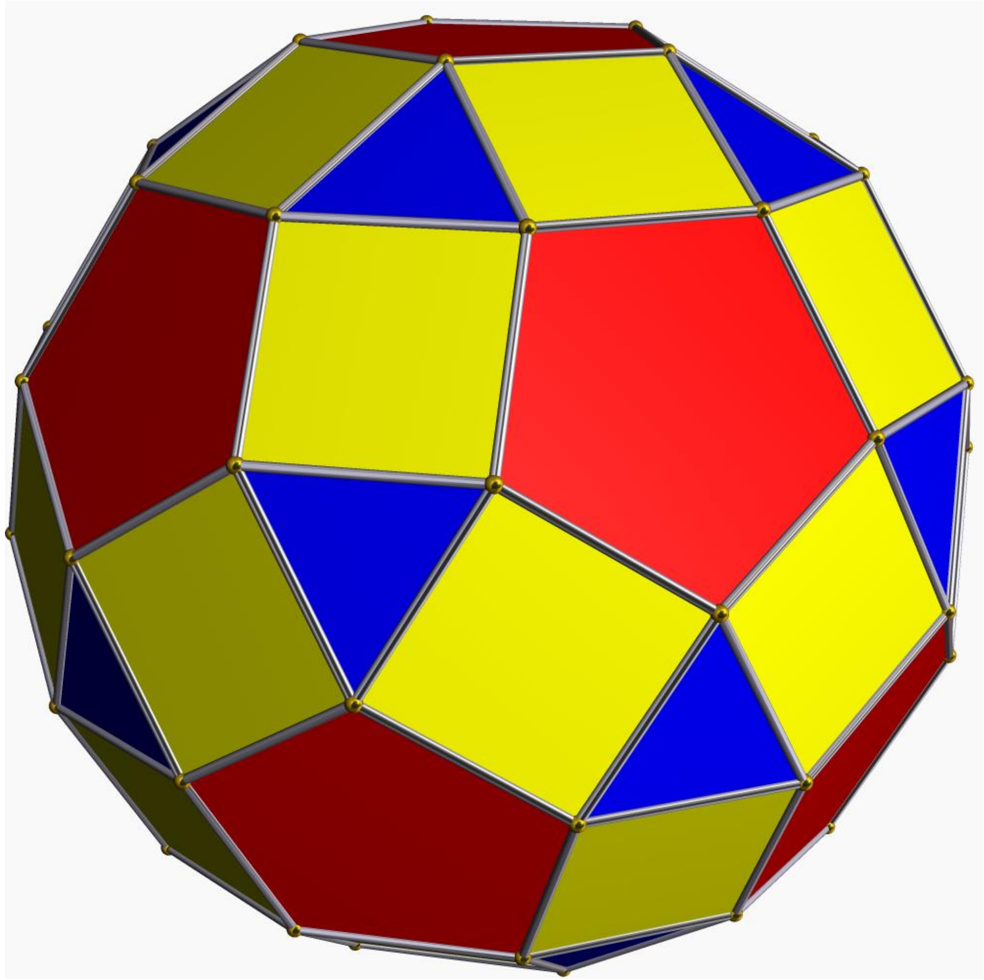
8	9	10	6	7	3	5	1	4	2
7	8	9	10	4	5	6	2	1	3
10	7	8	9	5	6	1	3	2	4
9	10	5	8	6	7	2	4	3	1
6	5	1	7	3	2	4			
5	6	7	4	2	1	3			
3	2	6	5	1	4				
2	4	3	1						
1	3	4	2						
4	1	2	3						

Chow)

YOU EITHER KNOW THIS, OR YOU'LL LEARN SOMETHING!

The pictured polyhedron has 60 faces and 62 vertices.

How many edges does it have?



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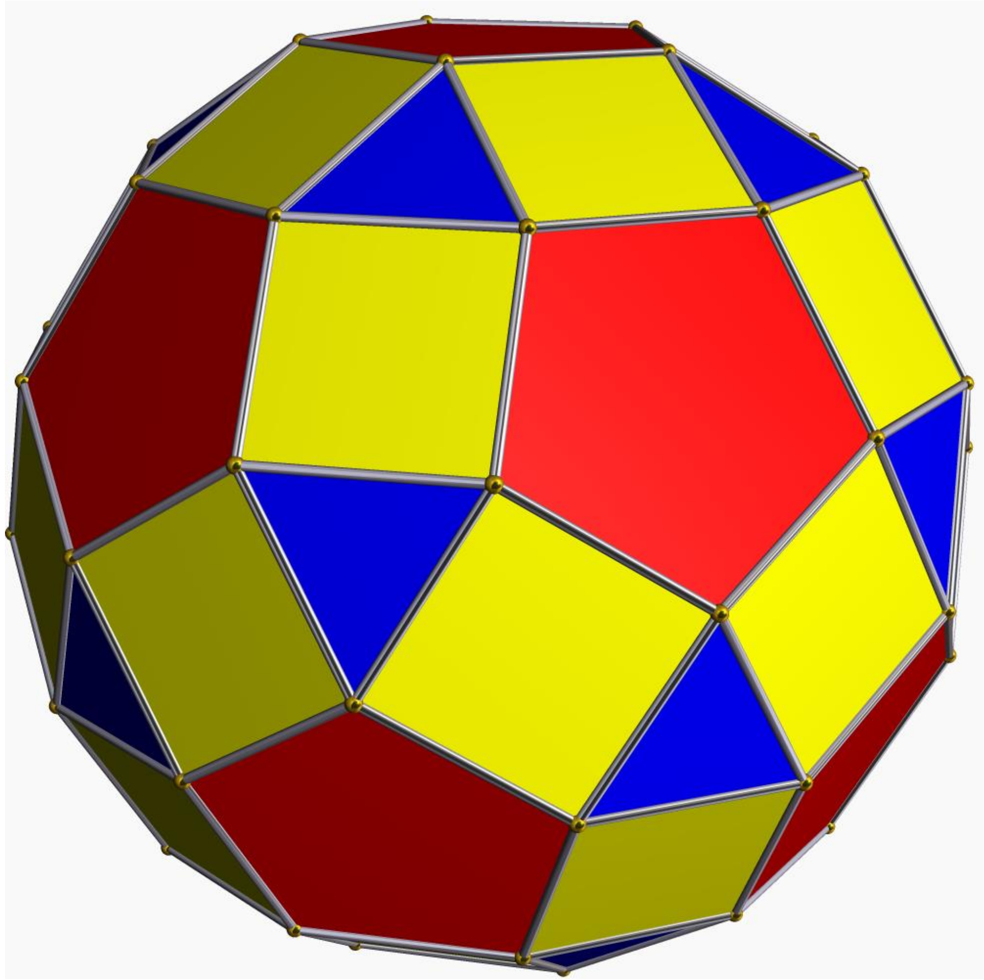
Euler's formula: $V - E + F = 2$.



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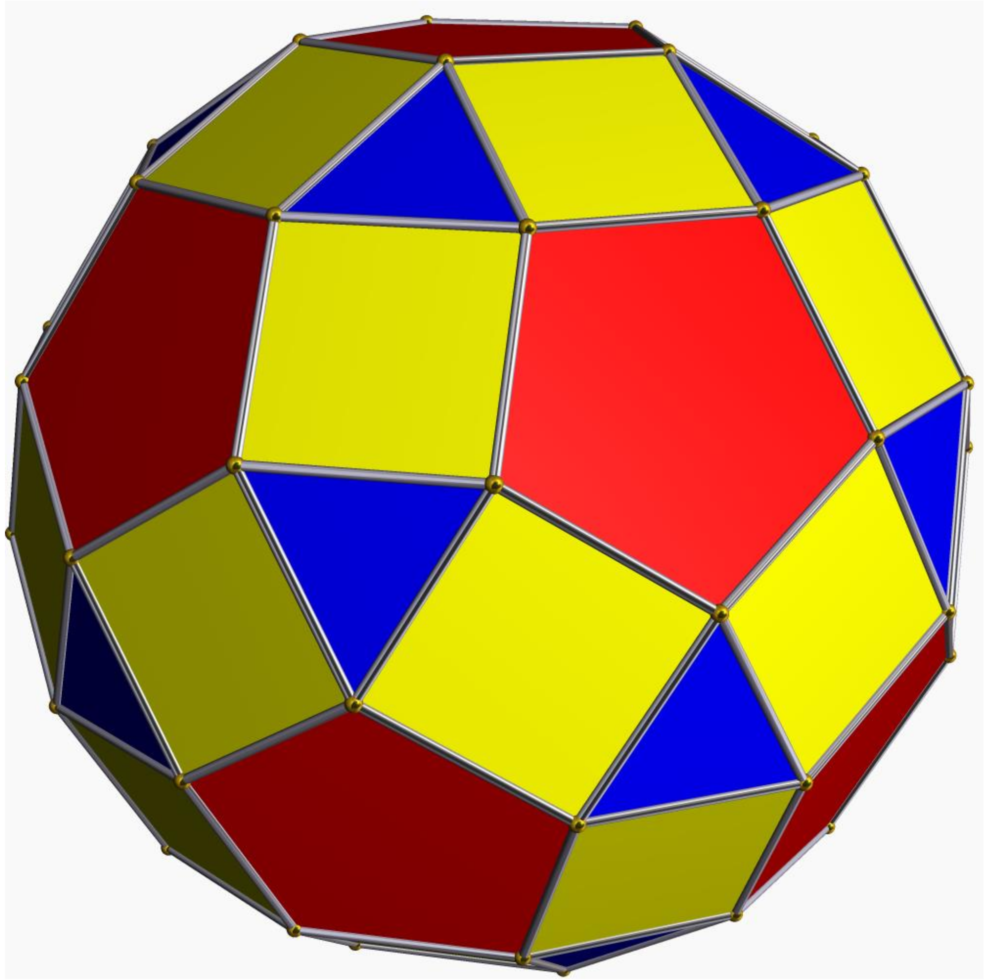


120

DO YOU KNOW HOW TO PROVE EULER’S FORMULA?

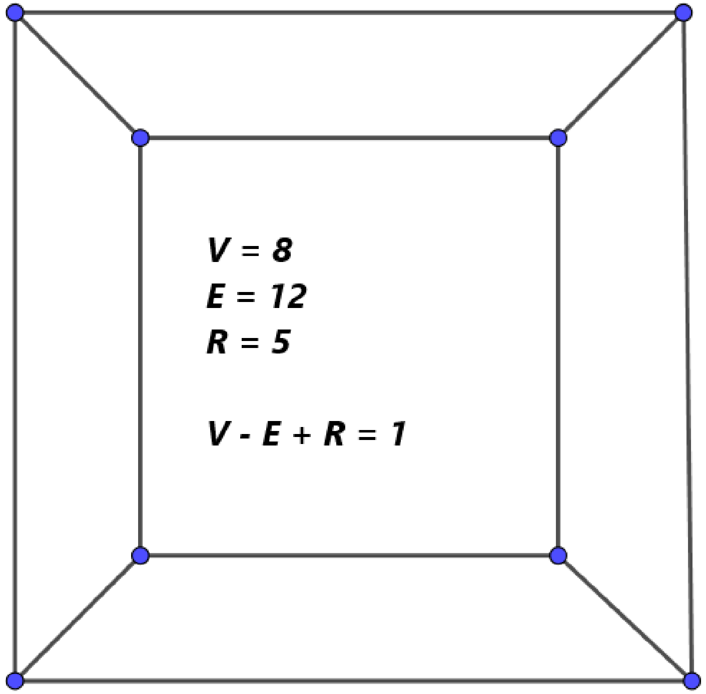
EULER'S FORMULA FOR GRAPHS

If we remove one side of a convex polyhedron, we can flatten it an open it up to a connected graph.

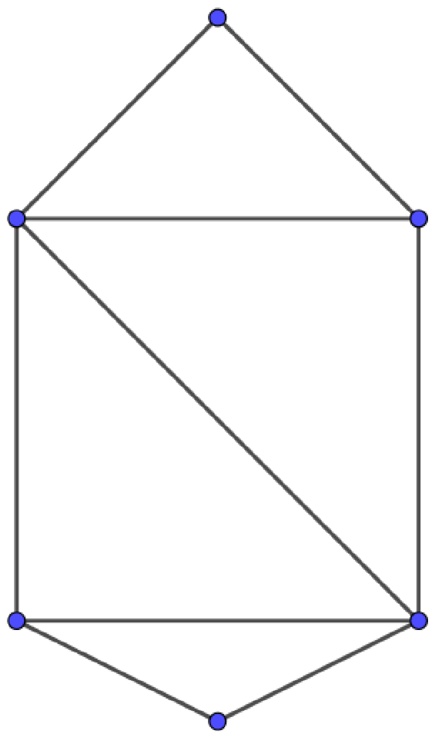
The faces become bounded regions, but there is one face missing. So, with V, E as before, R the number of regions, Euler's formula for a connected graph is

$$V - E + R = 1.$$

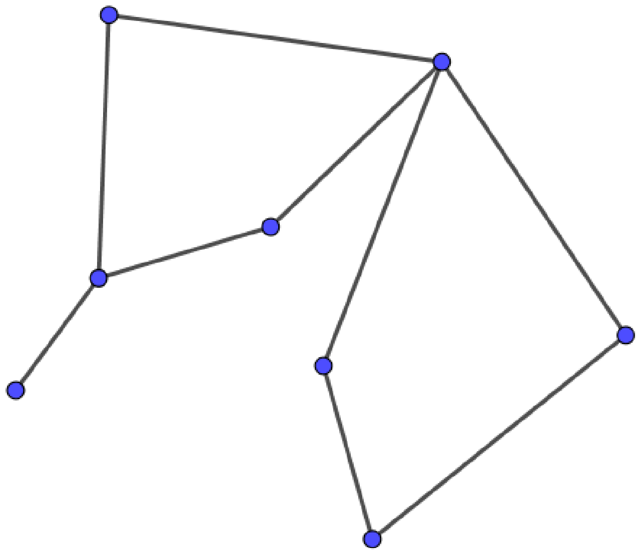
A squashed cube



GLAMOROUS GRAPHS



6 vertices, 9 edges, 4 regions



8 vertices, 9 edges, 2 regions

If a connected graph has 120 vertices and the *order* of each vertex Is 3.

How many regions does it have?

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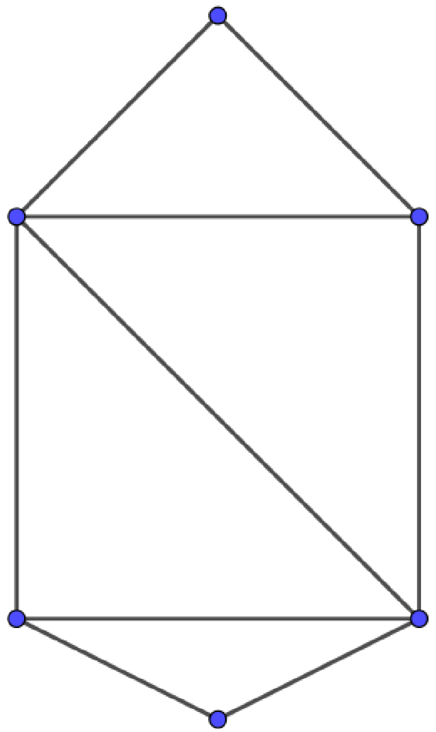

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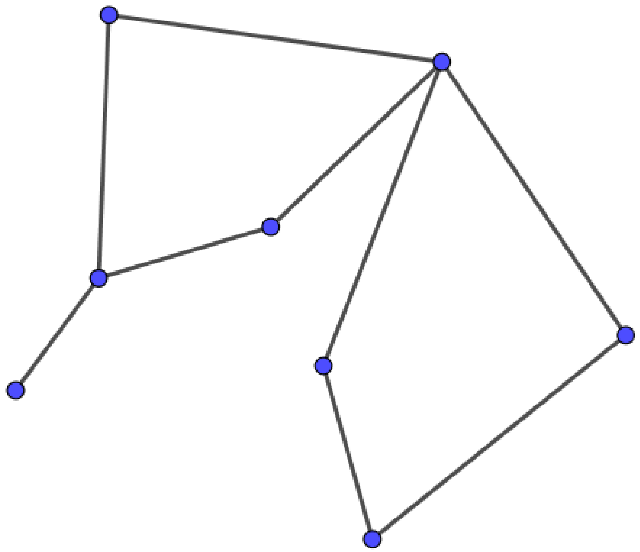

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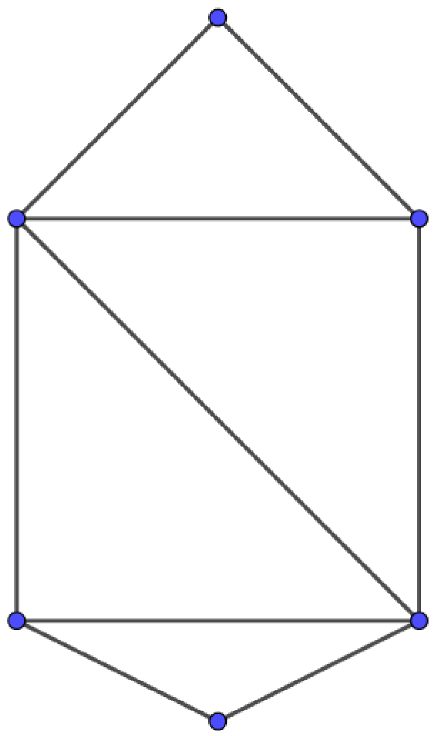

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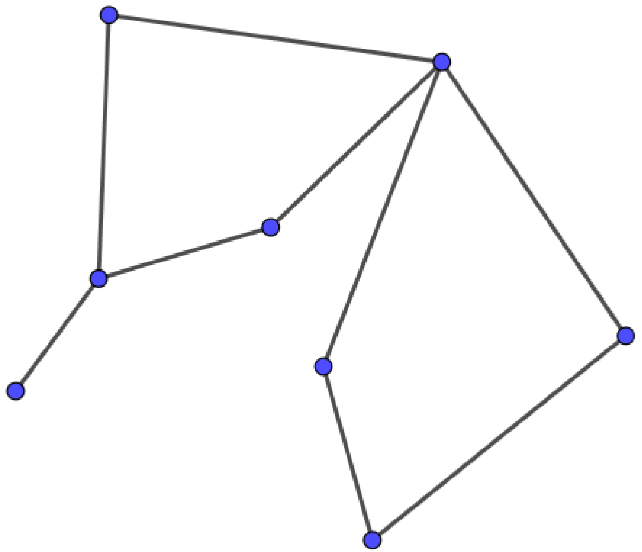


8 vertices, 9 edges, 2 regions

If a connected graph has 120 vertices and the *order* of each vertex Is 3.

How many regions does it have?

Solution: For every vertex there are 3 edges. That would be $3V$ edges, except that this counts them twice; each edge has two vertices.. So, $E = \frac{3}{2}V = 180$.

$R = 1 + E - V = 61$

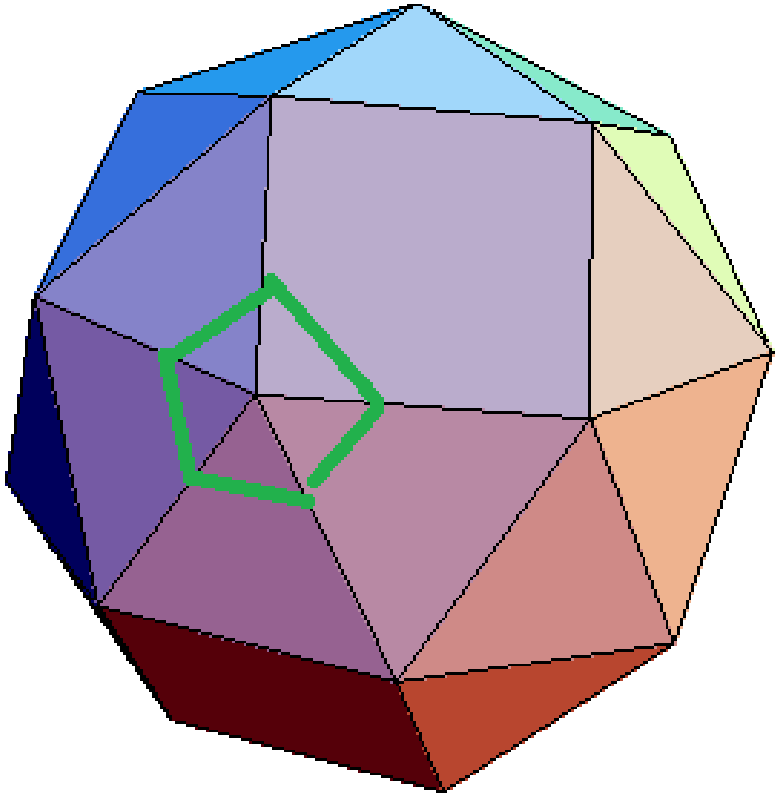
A TOUGH TOUGHIE ?

Suppose P is a convex polyhedron. Get a new polyhedron Q by cutting off a tip from each vertex. Suppose Q has V vertices, E edges, and F faces, and one of V, E, F equals 1001. How many edges does P have?

(From *Bicycle or Unicycle*,

by

D. Velleman and S. Wagon)



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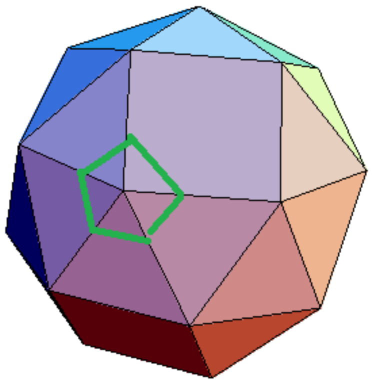

SOLUTION: Let us call V_0, E_0, F_0 the number of vertices, edges, faces, respectively of P. Our task is to find E_0.
Consider a vertex of P and suppose there are n edges ending at that vertex. That means that in the transition from P to Q, there will be n edges **added**. If we take the sum of all the $n's$ for all the vertices of P we get $2E_0$. We see that $E = E_0 + 2E_0 = 3E_0$, a multiple of 3. Thus $E \neq 1001$.
In moving from P to Q every vertex of P gets replaced by n vertices, so V is the sum of all the $n's$ for all the vertices of $P, which\ was\ 2E_0$. That is $V = 2E_0$, an even number. So $V \neq 1001$. This forces $F = 1001$.
Now we use Euler's equation on all we found:

$$2 = V - E + F = 2E_0 - 3E_0 + 1001 = 1001 - E_0.$$

Thus

$$E_0 = 1001 - 2 = 999.$$

EULER AND SOCCER

- Soccer balls are made by stitching together pentagonal and hexagonal pieces, with three pieces meeting at each vertex. If such a ball is made using p pentagonal pieces and h hexagonal pieces, what is the answer to the following questions.

A. True or False? There could be any number of pentagonal pieces.

B. The number of pentagonal pieces must always be the same and it equals _______?

C. True or False? There could be any number of hexagonal pieces.

D. The number of hexagonal pieces must always be the same and it equals _____ ?

EULER AND SOCCER

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A. True or False? There could be any number of pentagonal pieces. F

B. The number of pentagonal pieces must always be the same and it equals 12?

C. True or False? There could be any number of hexagonal pieces. T

D. The number of hexagonal pieces must always be the same and it equals _____ ?

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EULER AND SOCCER - EXPLANATIONS

- Let us determine V, E, F. The total number of vertices of the faces is $5p + 6h$.
- At each vertex of the football three of these come together so $V = \frac{1}{3}(5p + 6h)$.
- Since there are three edges at each vertex, at each edge has two vertices, $E = \frac{3}{2}V = \frac{1}{2}(5p + 6h)$.
- Of course, $F = p + h$.
- Euler's formula gives

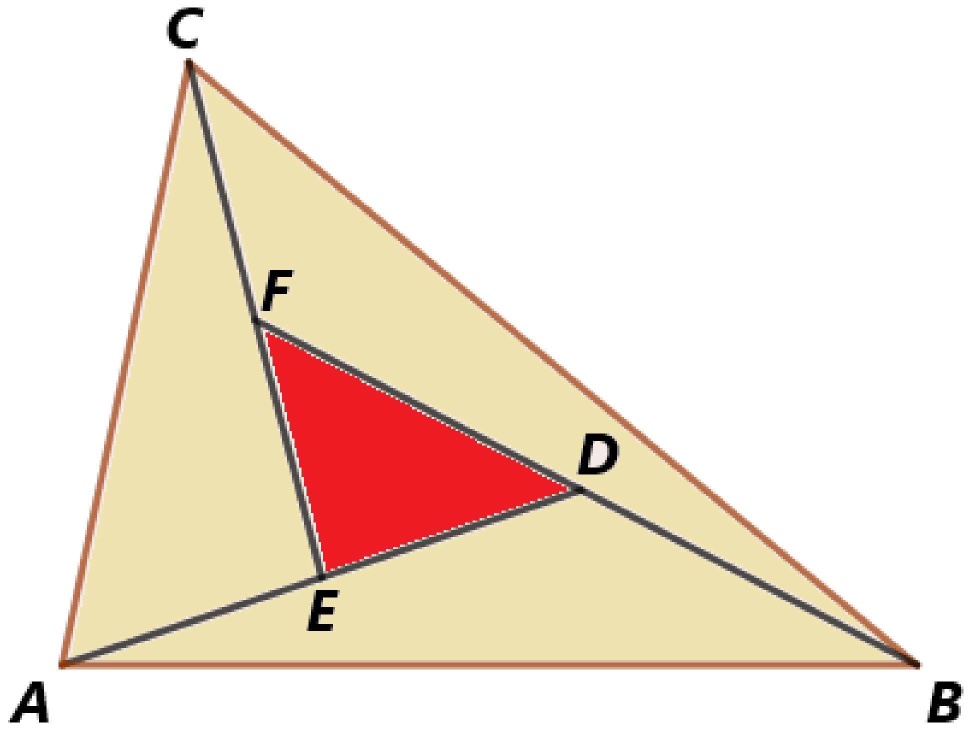
$$2 = V - E + F = \frac{1}{3}(5p + 6h) - \frac{1}{2}(5p + 6h) + p + h = \frac{p}{6}.$$

It follows that $p = 12$ and there is no restriction on h.

FROM LAST TIME: TRIANGULAR TRIANGULATIONS

- Segments are drawn in triangle ABC in such a way that D is the midpoint of BF, E is the midpoint of AD, and F is the midpoint of CE.

- If the area of triangle ABC is 1, what is the area of triangle DEF?

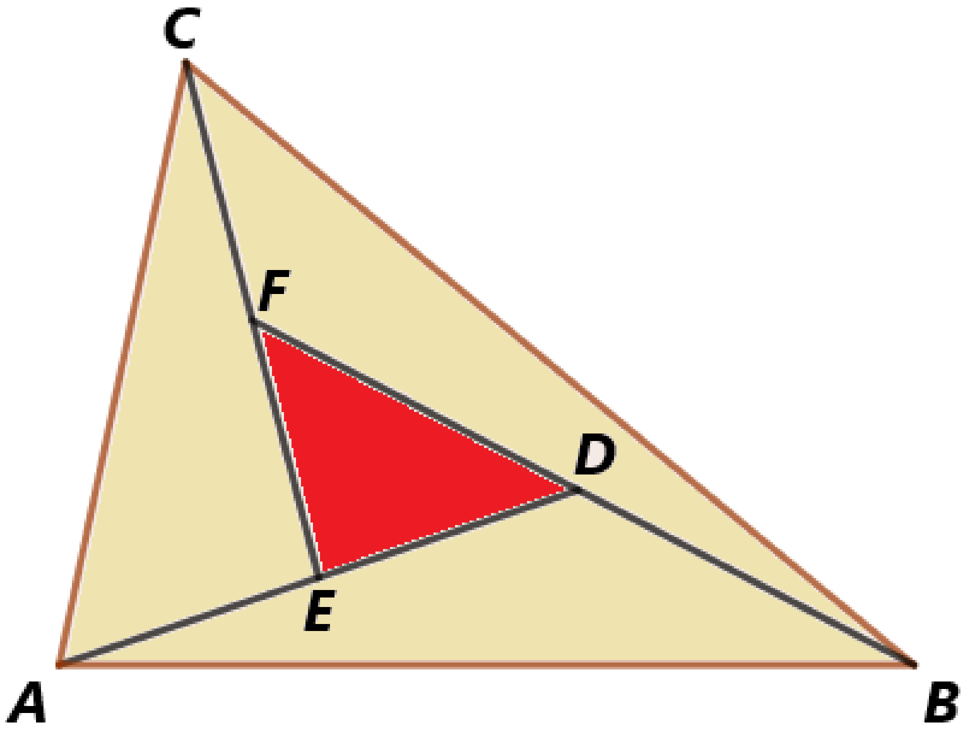


TRIANGULAR TRIANGULATIONS – A HINT

- Segments are drawn in triangle ABC in such a way that D is the midpoint of BF, E is the midpoint of AD, and F is the midpoint of CE.

- If the area of triangle ABC is 1, what is the area of triangle DEF?

  Hint: Relate the area of triangle ABC to the areas of the four surrounding triangles.



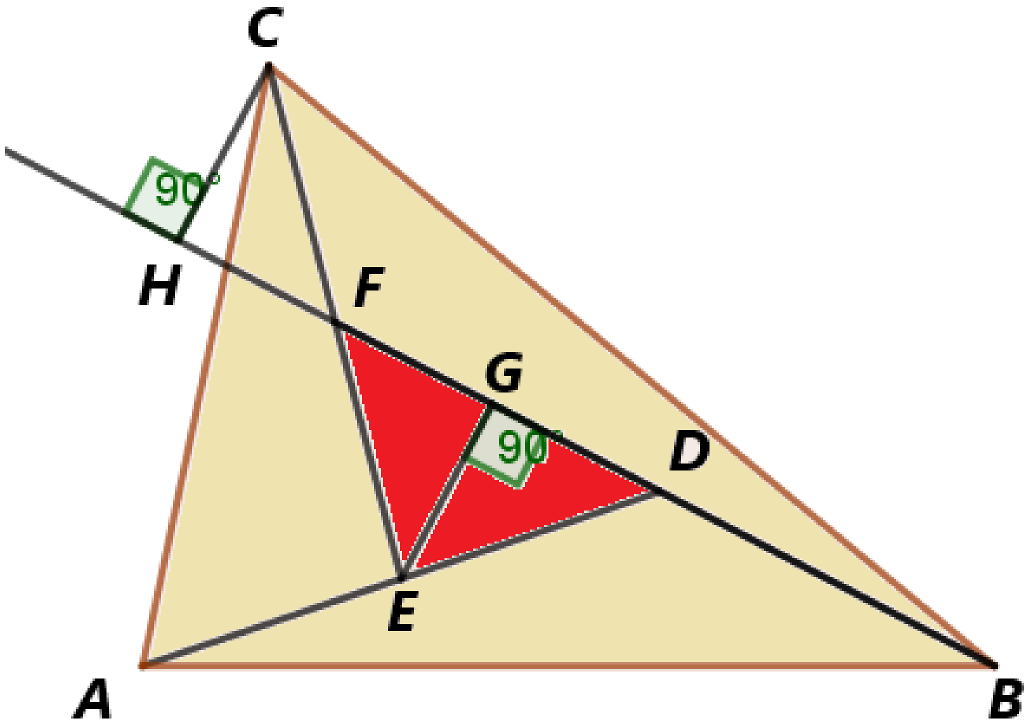
TRIANGULAR TRIANGULATIONS – ANOTHER HINT

- Segments are drawn in triangle ABC in such a way that D is the midpoint of BF, E is the midpoint of AD, and F is the midpoint of CE.

- If the area of triangle ABC is 1, what is the area of triangle DEF?

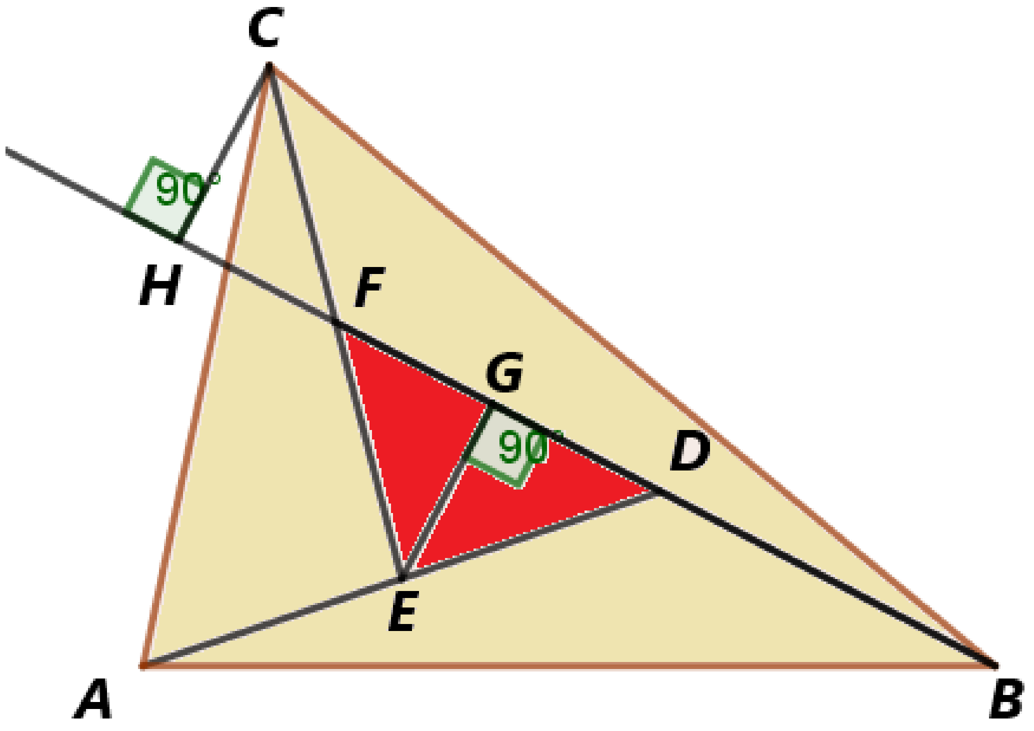
  Hint: Relate the area of triangle ABC to the areas of the four surrounding triangles.

  Does this picture help in relating $[DEF\}$ and $[BCF]$?



TRIANGULAR TRIANGULATIONS – SOLUTION

- Segments are drawn in triangle ABC in such a way that D is the midpoint of BF, E is the midpoint of AD, and F is the midpoint of CE.
- If the area of triangle ABC is 1, what is the area of triangle DEF?
- **Solution:** Triangles EFG and FCH are easily seen to be congruent, thus $|EG| = |CH|$. Thus
- $[BCF] = \frac{1}{2}|BF| \cdot |CH| = \frac{1}{2} \cdot 2|DF| \cdot |EG| = 2[DEF]$.
- Similarly, $[ACE] = 2[DEF], [ADB] = 2[DEF]$.



$1 = [ABC]$=[ACE]+[ADB]+[BCF]+[DEF] = 7[DEF]. We get $[DEF\} = \frac{1}{7}$.