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AN ALGORITHMIC APPROACH TO LATTICES AND ORDER IN DYNAMICS

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Abstract. Recurrent versus gradient-like behavior in global dynamics can be characterized via a surjective lattice homomorphism between certain bounded, distributive lattices, that is, between attracting blocks (or neighborhoods) and attractors. Using this characterization, we build finite, combinatorial models in terms of surjective lattice homomorphisms, which lay a foundation for a computational theory for dynamical systems that focuses on Morse decompositions and index lattices. In particular we present an algorithm that builds a combinatorial model that represents a Morse decomposition for the underlying dynamics. We give computational examples that illustrate the theory for both maps and flows.

1. **Introduction.** Recent work in computational dynamics has led to the use of combinatorial representations of dynamical systems to extract rigorous statements about global dynamics and how this dynamics changes with respect to parameters, cf. [16, 2, 1, 5, 8]. Further development of these methods relies on understanding the way in which lattices and order naturally play a role in dynamics on the fundamental level of attractors, repellers, and invariant sets, cf. [19]. Analogues of these basic concepts in dynamical systems theory also exist in directed graphs and are used to analyze combinatorial representations of dynamical systems. Recent results have addressed how robustly these combinatorial representations may capture the global dynamics of an underlying dynamical system, cf. [20, 21]. In this paper we present an algorithmic framework to determine when a specific combinatorial representation captures the underlying dynamical structure. These structures provide a natural framework for the development of computational algorithms.

For clarity in this introduction we consider the setting of a discrete time dynamical system generated by iteration of a continuous map $f\colon X \to X$. However, the results can also be applied in the continuous time setting, cf. [20] and the examples in Section 6. The most significant assumption we make is that X is a compact metric space. We emphasize that we do *not* assume that f is injective nor surjective.

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1.1. **Lattice structures in dynamics.** The set of attractors $\mathsf{Att}(f)$ in a dynamical system has a natural distributive lattice structure, as does the set of attracting blocks $\mathsf{ABlock}(f)$. A subset $N \subset X$ is an *attracting block* if $f(\mathrm{cl}\, N) \subset \mathrm{int}\, N$, and $\omega(N) = A$ is the associated attractor, cf. [19, 20]. In many cases attracting blocks are readily computable, while the lattice of attractors is not directly computable in general. For computational purposes, in place of $\mathsf{ABlock}(f)$ the more restrictive lattice of regular closed, attracting blocks $\mathsf{ABlock}_{\mathscr{R}}(f)$ may be considered. The latter is a sublattice of the Boolean algebra $\mathscr{R}(X)$ of regular closed sets in X with the binary operations $\vee = \cup$ and $\wedge = \mathrm{cl}\,(\mathrm{int}\,(\cdot \cap \cdot))$. The definitions of all of these structures and their properties are given in detail in [19, 20] to which we refer the reader, see also Section 1.2. We emphasize that a fundamental consideration when working with the above structures is that, while the lattice operations on $\mathsf{ABlock}_{\mathscr{R}}(f)$ are join and meet in $\mathscr{R}(X)$, the lattice operations on $\mathsf{Att}(f)$ are $\vee = \cup$ and $\wedge = \omega(\cdot \cap \cdot)$, where ω denotes the operation of taking the ω-limit set.

The global structure of dynamics in terms of separating gradient-like and recurrent behavior is captured by the lattice epimorphism

$$\omega\colon \mathsf{ABlock}_{\mathscr{R}}(f) \twoheadrightarrow \mathsf{Att}(f). \tag{1}$$

A choice of finite sublattices of $\mathsf{N} \subset \mathsf{ABlock}_{\mathscr{R}}(f)$ and $\mathsf{A} \subset \mathsf{Att}(f)$ with $\omega\colon \mathsf{N} \twoheadrightarrow \mathsf{A}$ may be regarded as a finite rendering of the global dynamics of a system. The sublattice A is referred to as an *attractor lattice*, and N is called an *index lattice* for A. The traditional terminology for index lattice is *index filtration*, cf. [9]. However, the latter are not filtrations per se which justifies the terminology index *lattice*.

The associated commutative diagram is given by

$$\begin{array}{ccc} \mathsf{N} & \xrightarrow{\subset} & \mathsf{ABlock}_{\mathscr{R}}(f) \\ \Big\downarrow{\scriptstyle\omega} & & \Big\downarrow{\scriptstyle\omega} \\ \mathsf{A} & \xrightarrow{\subset} & \mathsf{Att}(f) \end{array} \tag{2}$$

From an index lattice one traditionally extracts a Morse decomposition, which is an alternative, equivalent description of the dynamical information contained in $\omega\colon \mathsf{N} \twoheadrightarrow \mathsf{A}$.

1.1 Definition A *Morse decomposition* is an order embedding $\pi\colon \mathsf{M} \hookrightarrow \mathsf{P}$, where M and P are finite posets and M consists of nonempty, compact, pairwise disjoint invariant sets $M \subset X$ of f such that for every complete orbit γ_x through a point $x \in X \setminus \cup_{\mathsf{M}} M$ there exist $p, p' \in \mathsf{P}$ with $p < p'$ such that

$$\omega(x) \subset \pi^{-1}(p) \quad \text{and} \quad \alpha_{\mathrm{o}}(\gamma_x^-) \subset \pi^{-1}(p'). \tag{3}$$

∎

In [6] Conley introduced Morse decompositions to describe the dichotomy between gradient-like and recurrent behavior, which is central to the understanding of the global dynamics of a system.

To obtain a Morse decomposition from $\omega\colon \mathsf{N} \twoheadrightarrow \mathsf{A}$ we proceed as follows. Note that every lattice is also a poset via $a \leqslant b$ if $a \vee b = b$. The partial orders on the lattices $\mathsf{Att}(f)$ and $\mathsf{ABlock}_{\mathscr{R}}(f)$ correspond to set inclusion. The join-irreducible elements of a lattice are those that have exactly one immediate predecessor in the partial order of the lattice; given such an element c we denote its (unique) predecessor by $\overleftarrow{c}$. Extracting the join-irreducible elements of a finite distributive lattice defines the contravariant functor J from the category of finite distributive lattices to the category of finite posets, cf. Section 4. Due to this functoriality the lattice epimorphism $\mathsf{N} \twoheadrightarrow \mathsf{A}$ yields an order embedding $\mathsf{J}(\mathsf{A}) \hookrightarrow \mathsf{J}(\mathsf{N})$, where $\mathsf{J}(\mathsf{A})$ and $\mathsf{J}(\mathsf{N})$ are posets with respect to set inclusion. The posets $\mathsf{J}(\mathsf{A})$ and $\mathsf{J}(\mathsf{N})$ have dynamical meaning through their representations as invariant sets and regular closed isolating blocks respectively. The tool for constructing these representations is the *Conley form*. We summarize the relevant properties here and refer the reader to [21] for details.

The Conley form is defined in terms of a duality map. The dual of an attractor is given by $A^* := \{x \in X \mid \omega(x) \cap A = \varnothing\}$ and the dual of a regular closed attracting block is given by $N^\# := \mathrm{cl}\,(X \cap N^c)$. Then we obtain injective maps $\mathsf{J}(\mathsf{A}) \to \mathsf{Invset}(X)$ and $\mathsf{J}(\mathsf{N}) \to \mathscr{R}(X)$ given by

$$A \mapsto A \wedge \overleftarrow{A}^* = A \cap \overleftarrow{A}^* \quad \text{and} \quad N \mapsto N \wedge \overleftarrow{N}^\# = \mathrm{cl}\,(N \cap \overleftarrow{N}^c).$$

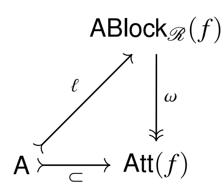
whose images

$$\mathsf{M}(\mathsf{A}) := \{M = A \wedge \overleftarrow{A}^* \mid A \in \mathsf{J}(\mathsf{A})\} \quad \text{and} \quad \mathsf{T}(\mathsf{N}) := \{T = N \wedge \overleftarrow{N}^\# \mid N \in \mathsf{J}(\mathsf{N})\} \tag{4}$$

are posets with partial orders induced by $\mathsf{J}(\mathsf{A})$ and $\mathsf{J}(\mathsf{N})$ respectively. The poset $\mathsf{M}(\mathsf{A})$ is called the *Morse representation* of the attractor lattice A, and $\mathsf{T}(\mathsf{N})$ is referred to as a *Morse tiling* of the phase space X. In [21] we show that the order-embedding $\pi\colon \mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N})$ is a Morse decomposition in the sense of Definition 1.1, which is referred to as a *tessalated* Morse decomposition. In particular $\mathsf{M}(\mathsf{A})$ is a collection of invariant sets, and $\mathsf{T}(\mathsf{N})$ is a collection of regular closed isolating blocks.

The following theorem answers the question of existence of index lattices for every (finite) attractor lattice $\mathsf{A} \subset \mathsf{Att}(f)$.

1.2 Theorem (cf. [19, 20]) For every finite sublattice $\mathsf{A} \subset \mathsf{Att}(f)$, there exists a lattice monomorphism ℓ such that the following diagram commutes:



∎

A map ℓ as in the above diagram is called a *lift* of the attractor lattice A to regular closed attracting blocks and $\mathsf{N} = \ell(\mathsf{A})$ is the associated index lattice of attracting blocks. This result implies that there is no fundamental obstruction to identifying a finite sublattice of attractors in a system by a corresponding sublattice of attracting blocks. The existence of the lift $\ell(\mathsf{A}) = \mathsf{N}$ so that $\mathsf{A} = \omega(\mathsf{N})$ in Theorem 1.2 is dually equivalent to the existence of an isomorphic, tesselated Morse decomposition $\pi\colon \mathsf{M}(\mathsf{A}) \leftrightarrow \mathsf{T}(\mathsf{N})$. In the context of flows, Theorem 1.2 is proved by Franzosa in [9], see also [10, 23]. Tesselated Morse decompositions and index lattices are fundamental building blocks in the theory of connection and transition matrices.

The finite structures in Diagram (2) provide a description of dynamics within a given resolution but are not directly computable in general [4]. To develop an algorithmic framework for the computation of these structures, we consider a *combinatorial model* for

$\omega\colon \mathsf{ABlock}_{\mathscr{R}}(f) \twoheadrightarrow \mathsf{Att}(f)$, which consists of two lattice homomorphisms: a lattice epimorphism $h\colon \mathsf{K} \twoheadrightarrow \mathsf{L}$ of finite distributive lattices, called the *interior homomorphism*, and a lattice monomorphism $e\colon \mathsf{K} \rightarrowtail \mathsf{ABlock}_{\mathscr{R}}(f)$, called the *evaluation homomorphism*, which links the combinatorial model to the dynamical system through the diagram

$$\begin{array}{ccc} \mathsf{K} & \overset{e}{\rightarrowtail} & \mathsf{ABlock}_{\mathscr{R}}(f) \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle \omega} \\ \mathsf{L} & & \mathsf{Att}(f) \end{array} \tag{5}$$

The objective is to perform computations on combinatorial models and translate the result to the underlying system through the evaluation homomorphism. To clearly illustrate the concept of a combinatorial model, we describe the well-studied approach of outer approximation. We emphasize that outer approximation is just one method of constructing a combinatorial model, but when feasible, such a model has additional properties as described below. In Section 5 we outline a general characterization of combinatorial models, and in Section 6 we consider other methods for constructing combinatorial models for flows as well as maps.

1.2. **Outer approximation.** In [16, 2, 1, 5, 8] a computational method for global dynamics is developed which builds a combinatorial representation. Such a representation is based on a finite discretization of the phase space X by a *grid*, which is defined as a finite subalgebra of $\mathscr{R}(X)$, the regular closed subsets of X, such as a triangulation or a cubical grid when X is a region in $\mathbb{R}^n$, see [20]. We denote an indexing set for the grid by $\mathcal{X}$. In particular, given $\xi \in \mathcal{X}$ the corresponding grid element is denoted by $|\xi| \in \mathscr{R}(X)$. The *evaluation map* $|\cdot| : \mathsf{Set}(\mathcal{X}) \to \mathscr{R}(X)$ is extended to subsets of $\mathcal{X}$ by

$$|\mathcal{U}| := \bigcup_{\xi\in\mathcal{U}} |\xi| \,.$$

By Corollary 3.6 in [20], this map is a lattice homomorphism.

The map $f\colon X \to X$ is approximated by a relation $\mathcal{F}$ on the set of grid elements $\mathcal{X}$. In general we denote a relation f on a set X by (X, f). Note that a dynamical system generated by iterating the map $f\colon X \to X$ is a relation (X, f) given by $\{(x, f(x)) \in X \times X\}$. Similarly the relation $(\mathcal{X}, \mathcal{F})$ can be viewed as a multivalued map defined by $\mathcal{F}(\xi) := \{\eta \in \mathcal{X} \mid (\xi, \eta) \in \mathcal{F}\}$.

1.3 Definition (cf. [16, 26]) Let $f\colon X \to X$ be a continuous map and let $\mathcal{X}$ be the indexing set for a grid on X. A relation $(\mathcal{X}, \mathcal{F})$ is an *outer approximation* of a dynamical system (X, f) if

$$f(|\xi|) \subset \operatorname{int} |\mathcal{F}(\xi)| \text{ for all } \xi \in \mathcal{X}. \tag{6}$$

■

Attractors for $\mathcal{F}$ are defined as sets satisfying $\mathcal{F}(\mathcal{A}) = \mathcal{A}$ and form the lattice $\mathsf{Att}(\mathcal{F})$ under the operations $\vee = \cup$ and $\wedge = \omega(\cdot \cap \cdot)$ where the ω-limit set is defined in equation (14) in Section 3.1. Subsets $\mathcal{U}$ for which $\mathcal{F}(\mathcal{U}) \subset \mathcal{U}$ are called *forward invariant* sets and form the lattice $\mathsf{Invset}^+(\mathcal{F})$ under the operations of union and intersection. The properties of these lattices as well as the epimorphism $\omega\colon \mathsf{Invset}^+(\mathcal{F}) \twoheadrightarrow \mathsf{Att}(\mathcal{F})$ are described in Section 3 and [20]. We obtain the commutative diagram

$$\begin{array}{ccccc}
\mathsf{Invset}^+(\mathcal{F}) & \xleftrightarrow{|\cdot|} & \mathsf{N} & \xrightarrow{\subset} & \mathsf{ABlock}_{\mathscr{R}}(f) \\
\downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\omega} \\
\mathsf{Att}(\mathcal{F}) & \xrightarrow{\omega(|\cdot|)} & \mathsf{A} & \xrightarrow{\subset} & \mathsf{Att}(f)
\end{array} \tag{7}$$

Hence an outer approximation determines a combinatorial model for which $\omega\colon \mathsf{Invset}^+(\mathcal{F}) \twoheadrightarrow \mathsf{Att}(\mathcal{F})$ is the interior homomorphism. In this case, a *connecting homomorphism* between $\mathsf{Att}(\mathcal{F})$ and A exists, and so this is an example of a *commutative combinatorial model* as described in Section 5.

Combinatorial models as described above contain computable information about the underlying system. If we dualize Diagram (7), then we obtain the tesselated Morse decomposition $\pi\colon \mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{RC}(\mathcal{F}) \hookrightarrow \mathsf{SC}(\mathcal{F}) \longleftrightarrow \mathsf{T}(\mathsf{N})$, cf. Diagram (21), where $\mathsf{SC}(\mathcal{F})$ and $\mathsf{RC}(\mathcal{F})$ are the posets of strongly connected components and recurrent (cyclic) strongly connected components of $\mathcal{F}$ respectively, cf. Section 3.2. In general the tesselation $\mathsf{T}(\mathsf{N}) \cong \mathsf{SC}(\mathcal{F})$ is a very large set, while $\mathsf{RC}(\mathcal{F})$ is relatively small in size.

Two central questions arise.

(i) Can every sublattice $\mathsf{A} \subset \mathsf{Att}(f)$ be realized via an outer approximation $\mathcal{F}$ as in Diagram (7) ?

(ii) Given $\mathcal{F}$, do coarser tesselations exist, for example tesselated Morse decompositions of the form $\mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{RC}(\mathcal{F}) \longleftrightarrow \mathsf{T}(\mathsf{N})$?

In principle, the second question involves a finite calculation, since $\mathcal{F}$ is fixed, except that A may not be known. However, we show in Section 5 that one can algorithmically check whether coarser tesselations exist, but perhaps the attractor sublattice of this coarser tesselation is itself also coarser. If A is not known a priori, then allowing A to be coarsened in order to obtain a much coarser tesselation is of no practical concern.

The first question can be answered in the context of the lifting problem in Theorem 1.2. Convergence of a sequence $\mathcal{F}_n$ of outer approximations on grids $\mathcal{X}_n$ corresponds to both the diameters of the grid elements in $\mathcal{X}_n$ and the errors in images of grid elements under $\mathcal{F}_n$ to tend to zero as $n \to \infty$. If we consider appropriate convergent sequences of outer approximations via consecutive refinement, called a convergent cofiltrations, we have the following result.

1.4 Theorem (Theorem 1.2 in [20]) Let $f : X \to X$ be a continuous map on a compact metric space X. Let $(\mathcal{X}_n, \mathcal{F}_n)$ be a convergent cofiltration of outer approximations. Then for every finite sublattice $\mathsf{A} \subset \mathsf{Att}(f)$ there exists $n_{\mathsf{A}} \in \mathbb{N}$ such that for all $n \geqslant n_{\mathsf{A}}$ there exists a lift $\ell_n : \mathsf{A} \rightarrowtail \mathsf{Invset}^+(\mathcal{F}_n)$ of the inclusion map $\mathsf{A} \rightarrowtail \mathsf{Att}(f)$ through

$\omega(|\cdot|)\colon \mathsf{Invset}^+(\mathcal{F}_n) \to \mathsf{Att}(f)$, i.e. the following diagram commutes

$$\begin{array}{ccc} & & \mathsf{Invset}^+(\mathcal{F}_n) \\ & \overset{\ell_n}{\nearrow} & \Big\downarrow \omega(|\cdot|) \\ \mathsf{A} & \xrightarrow[\subset]{} & \mathsf{Att}(f) \end{array}$$

■

Using Theorem 1.4 we set $\mathsf{N}_n := |\ell_n(\mathsf{A})| \cong \mathsf{A}$, which induces an isomorphic tesselated Morse decomposition $\pi_n\colon \mathsf{M}(\mathsf{A}) \leftrightarrow \mathsf{T}(\mathsf{N}_n)$ for all $n \geqslant n_{\mathsf{A}}$. This also provides an alternative proof for the existence of index lattices for Morse decompositions as given in Theorem 1.2.

The asymptotic lifting theorem answers both questions in (i) and (ii) with respect to existence. However, a fundamental issue still remains unanswered by this asymptotic result. If a computation is performed at a certain fixed resolution, how can we algorithmically determine whether a lift exists, and if so, construct a lift to obtain a coarser tesselated Morse decomposition. To develop such an algorithm, we make use of a generalization of the Birkhoff representation theorem, Theorems 1.5 and 1.6 below, to characterize the existence of a lift in terms of the existence of a certain map between the strongly connected components and the recurrent components of the directed graph defined by $\mathcal{F}$, and then algorithmically construct such a map. This is one of the main results in this paper and is proved in Theorem 2.4 of Section 2. Theorem 2.4 can be applied much more broadly, and in Section 5 we use this theorem to describe how to extract dynamics from general combinatorial models.

1.3. **Generalizations of the Birkhoff representation theorem.** For a partially ordered set P, a subset $I \subset \mathsf{P}$ is said to be a *down-set* of P if for every $p \in I$ we have $q \in I$ whenever $q \leqslant p$. The set of all down-sets of the poset P is a lattice, under the operations of $\vee = \cup$ and $\wedge = \cap$, and it is denoted by $\mathsf{O}(\mathsf{P})$. Given $p \in \mathsf{P}$, the set $\downarrow p := \{q \mid q \leqslant p\}$ is in $\mathsf{O}(\mathsf{P})$, and every down-set of P can be written as the union of such sets. The down sets of the form $\downarrow p$ for $p \in \mathsf{P}$ exactly make the set $\mathsf{J}(\mathsf{O}(\mathsf{P}))$, join irreducible elements of $\mathsf{O}(\mathsf{P})$.

The classical Birkhoff representation theorem states that every finite distributive lattice L may be represented as the lattice of down-sets of a finite poset $(\mathsf{P}, \leqslant)$, i.e. $\mathsf{L} \cong \mathsf{O}(\mathsf{P})$. Moreover, every finite poset P is isomorphic to the set of join-irreducible elements of a finite distributive lattice L, ordered with respect to set inclusion, i.e. $\mathsf{P} \cong \mathsf{J}(\mathsf{L})$, cf. Section 4. As a generalization of the Birkhoff representation theorem we show the following two theorems in Section 4.2.

1.5 Theorem (First generalized Birkhoff theorem) Let $h\colon \mathsf{K} \twoheadrightarrow \mathsf{L}$ be a lattice epimorphism between finite distributive lattices. Then there exists, up to transitive extension / reduction, condensation, and isomorphism, a unique binary relation $\mathcal{F}$ on a finite set $\mathcal{X}$ such that there exist isomorphisms $\mathsf{K} \approx \mathsf{Invset}(\mathcal{F})$ and $\mathsf{L} \approx \mathsf{Att}(\mathcal{F})$ and the following diagram commutes:

$$\begin{array}{ccc} \mathsf{Invset}^+(\mathcal{F}) & \longleftrightarrow & \mathsf{K} \\ \omega \Big\downarrow\!\!\downarrow & & \Big\downarrow\!\!\downarrow h \\ \mathsf{Att}(\mathcal{F}) & \longleftrightarrow & \mathsf{L} \end{array} \tag{8}$$

∎

Representation of finite binary relations is captured by the following theorem.

1.6 Theorem (Second generalized Birkhoff theorem) Let $\mathcal{F}$ be a finite binary relation on a finite point set $\mathcal{X}$. Then there exists, up to isomorphism, a unique lattice epimorphism $h\colon \mathsf{K} \twoheadrightarrow \mathsf{L}$ between finite distributive lattices such that there exist isomorphisms $\mathsf{J}(\mathsf{K}) \approx \mathsf{SC}(\mathcal{F})$ and $\mathsf{J}(\mathsf{L}) \approx \mathsf{RC}(\mathcal{F})$ and the following diagram commutes:

$$\begin{array}{ccc} \mathsf{SC}(\mathcal{F}) & \longleftrightarrow & \mathsf{J}(\mathsf{K}) \\ \subset \big\uparrow & & \big\uparrow \mathsf{J}(h) \\ \mathsf{RC}(\mathcal{F}) & \longleftrightarrow & \mathsf{J}(\mathsf{L}) \end{array} \tag{9}$$

∎

The uniqueness part in the above theorems can be made more precise as follows. Two binary relations $(\mathcal{X},\mathcal{F})$ and $(\mathcal{X}',\mathcal{F}')$ are equivalent if and only if there exist isomorphisms $\mathsf{SC}(\mathcal{F}) \cong \mathsf{SC}(\mathcal{F}')$ and $\mathsf{RC}(\mathcal{F}) \cong \mathsf{RC}(\mathcal{F}')$ and the following diagram commutes

$$\begin{array}{ccc} \mathsf{SC}(\mathcal{F}) & \longleftrightarrow & \mathsf{SC}(\mathcal{F}') \\ \subset \big\uparrow & & \big\uparrow \subset \\ \mathsf{RC}(\mathcal{F}) & \longleftrightarrow & \mathsf{RC}(\mathcal{F}') \end{array} \tag{10}$$

The binary relations in Theorem 1.5 are chosen up to the above equivalence. The precise isomorphisms in Diagrams (8), (9) and (10) are given in Section 4.

More information about the category of finite binary relations can be found in Appendix A. The advantage of the generalized Birkhoff representation theorem is that $\omega\colon \mathsf{Invset}^+(\mathcal{F}) \twoheadrightarrow \mathsf{Att}(\mathcal{F})$ may serve as combinatorial model, and via the evaluation map e such binary relations necessarily represent *weak* outer approximations for the dynamical system, cf. [20]. We discuss several examples of the application of these methods to dynamical systems in Section 6, where we illustrate the main results using computational results from specific systems. In Section 3 we discuss the dynamics of finite binary relations / directed graphs, which culminates in a generalization of the Birkhoff representation theorem in Section 4.

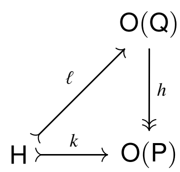
1.4. **Terminology and notation.** A bounded, distributive lattice $(\mathsf{L},\vee,\wedge)$ is a set L of objects with two binary operations join $\vee$ and meet $\wedge$ satisfying certain algebraic properties including the existence of a largest element 1 and a smallest element 0, cf. [7] and Section 2.1 of [19]. *All sublattices contain the neutral elements 0 and 1, and all lattice homomorphisms preserve the neutral elements 0 and 1.* The terms *lattice monomorphism* and *lattice epimorphism* refer to lattice homomorphisms which are injective and surjective respectively. The set of all (finite or) bounded, distributive lattices and lattice homomorphisms form a category. *Monic* morphisms in this category correspond to lattice monomorphisms, but not every *epic* morphism in this category is a lattice epimorphism. However, we do not make use of such categorical epic morphisms. We use the arrows $\rightarrowtail$ and $\twoheadrightarrow$ to indicate a lattice monomorphism and a lattice epimorphism respectively, and $\leftrightarrow$ denotes a lattice isomorphism.

The set of all finite posets and order-preserving maps forms a category. The terms *order injection* and *order surjection* are short for injective and surjective, order-preserving map respectively, and these are the monic and epic morphisms in the category respectively. An order-preserving map $\phi : \mathsf{P} \hookrightarrow \mathsf{Q}$ is an *order embedding* when $p \leqslant q$ if and only if $\phi(p) \leqslant \phi(q)$. Order embeddings, denoted by the arrow $\hookrightarrow$, are injective, but not every order injection, is an order embedding. We use $\twoheadrightarrow$ and $\leftrightarrow$ to denote order surjections and order isomorphisms respectively.

The operations defined by J and O are contravariant functors, as explained in detail in Section 4. The join functor J carries a lattice epimorphism to an order embedding and a lattice monomorphism to an order surjection. The *down-set functor* O carries an order surjection to a lattice monomorphism and an order embedding to a lattice epimorphism. These functors establish that the category of finite, distributive lattices is dually equivalent to the category of finite posets. This equivalence is called the Birkhoff representation theorem, Theorem 4.1.

2. **Lifts and Order Retractions.**

Let K, L be finite, distributive lattices with $h : \mathsf{K} \twoheadrightarrow \mathsf{L}$ a lattice epimorphism. By Birkhoff's representation theorem, cf. Theorem 4.1, we have $\mathsf{K} \cong \mathsf{O}(\mathsf{Q})$ and $\mathsf{L} \cong \mathsf{O}(\mathsf{P})$ for some finite posets P, Q. Suppose H is a sublattice of $\mathsf{O}(\mathsf{P})$. In this section we directly address the problem of algorithmically determining whether or not a lift ℓ of H to $\mathsf{O}(\mathsf{Q})$ exists as in the following diagram:



(11)

and if so, algorithmically constructing such a lift. Without loss of generality we may assume that $\mathsf{H} = \mathsf{O}(\mathsf{P})$, in which case a lift is a lattice monomorphism $\bar{\ell} : \mathsf{O}(\mathsf{P}) \rightarrowtail \mathsf{O}(\mathsf{Q})$ such that $h \circ \bar{\ell}$ is the identity map on $\mathsf{O}(\mathsf{P})$. Thus, if H is a proper sublattice of $\mathsf{O}(\mathsf{P})$, then the desired lift is $\ell = \bar{\ell} \circ k$.

Applying the join functor, $\mathsf{J}(h)\colon \mathsf{J}(\mathsf{O}(\mathsf{P})) \hookrightarrow \mathsf{J}(\mathsf{O}(\mathsf{Q}))$ is an order embedding. The latter induces the map $i\colon \mathsf{P} \hookrightarrow \mathsf{Q}$, which is given by the expression

$$i(p) = \min\{q \in \mathsf{Q} \mid p \in h(\downarrow q)\},$$

cf. Section 4. The functoriality of J implies that $\mathsf{J}(\bar{\ell}) \circ \mathsf{J}(h) = \mathrm{id}|_{\mathsf{J}(\mathsf{O}(\mathsf{P}))}$. Consequently, the existence of a lift ℓ corresponds to the existence of an order surjection $\sigma : \mathsf{Q} \twoheadrightarrow \mathsf{P}$ such that $\sigma \circ i$ is the identity on P, which is called an *order retraction of i*. In this case $\ell = \mathsf{O}(\sigma)$ is the desired lift. We now establish terminology and basic properties required for this construction.

2.1 Definition Let P, Q be finite posets and $i\colon \mathsf{P} \hookrightarrow \mathsf{Q}$ be an order embedding. An element $p \in \mathsf{P}$ is a *immediate successor in* P, or a P-*successor*, of $q \in \mathsf{Q}$ if $i(p) < q$ and whenever $i(p) \leqslant s < q$ for $s \in i(\mathsf{P})$ we have $i(p) = s$, i.e. there is no other element of $i(P)$ between $i(p)$ and q. Similarly, an element $r \in P$ is a *immediate predecessor in* P, or a P-*predecessor*, of $q \in \mathsf{Q}$ if $q < i(r)$ and whenever $q < s \leqslant i(r)$ for $s \in i(\mathsf{P})$ we have $s = i(r)$, i.e. there is no other element of $i(\mathsf{P})$ between q and $i(r)$. The set of all P-predecessors of q is denoted by $\mathrm{pred}_{\mathsf{P}}(q)$, and the set of all P-successors of q is denoted by $\mathrm{succ}_{\mathsf{P}}(q)$.

For brevity of notation, we may use only p to denote an element $p' = i(p) \in i(\mathsf{P})$ in places where it is clearly understood from context. In Figure 1 two examples of order embeddings are shown with $\mathrm{succ}_{\mathsf{P}}$ and $\mathrm{pred}_{\mathsf{P}}$ listed for each q. ▪

2.2 Lemma For any $q \in \mathsf{Q}\backslash i(\mathsf{P})$, $\mathrm{succ}_{\mathsf{P}}(q)$ and $\mathrm{pred}_{\mathsf{P}}(q)$ are antichains. ▪

Proof. Recall that a subset S of a poset is an antichain if $r \parallel s$ for every $r, s \in S$. For $r, s \in \mathrm{succ}_{\mathsf{P}}(q)$, we have $i(r) < q$ and $i(s) < q$. If $r \leqslant s$, then transitivity implies $i(r) \leqslant i(s) < q$. Since $r \in \mathrm{succ}_{\mathsf{P}}(q)$ and $i(s) \in i(\mathsf{P})$, we have $i(r) = i(s)$ by definition of P-successor. Hence $r = s$, since i is an order embedding, and $\mathrm{succ}_{\mathsf{P}}(q)$ is an antichain. Similarly, we can show that $\mathrm{pred}_{\mathsf{P}}(q)$ is an antichain. ∎

The following lemma establishes the fact that for an order retraction, $\sigma(q)$ must be either a P-successor or a P-predecessor of q or an element of P that lies between all the P-successors and P-predecessors of q.

2.3 Lemma Let $i : \mathsf{P} \hookrightarrow \mathsf{Q}$ be an order embedding. Suppose $\sigma : \mathsf{Q} \to \mathsf{P}$ is a retraction of i. Then σ is an order retraction of i if and only if σ is order-preserving on every chain contained in $\mathsf{Q}\backslash i(\mathsf{P})$ and

$$\sigma(q) \in \{p \in \mathsf{P} \mid \mathrm{succ}_{\mathsf{P}}(q) \subset \downarrow_{\mathsf{P}} p \text{ and } \mathrm{pred}_{\mathsf{P}}(q) \subset \uparrow_{\mathsf{P}} p\}. \tag{12}$$

for every $q \in \mathsf{Q}\backslash i(\mathsf{P})$. ▪

Proof. Suppose $\sigma : \mathsf{Q} \to \mathsf{P}$ is an order retraction of i. Then σ is order-preserving on all of Q, and for each $q \in \mathsf{Q}\backslash i(\mathsf{P})$ we must have $\mathrm{succ}_{\mathsf{P}}(q) \subset \downarrow_{\mathsf{P}} \sigma(q)$ and $\mathrm{pred}_{\mathsf{P}}(q)) \subset \uparrow_{\mathsf{P}} \sigma(q)$, because σ is order-preserving.

Conversely, suppose σ is a retraction of i satisfying the hypotheses of the lemma. We must show that σ is order-preserving on Q, and it suffices to show that for each $q \in \mathsf{Q}$, we have $\sigma(r) \leqslant \sigma(q)$ whenever $r \in \mathrm{succ}_{\mathsf{Q}}(q)$ and $\sigma(s) \geqslant \sigma(q)$ whenever $s \in \mathrm{pred}_{\mathsf{Q}}(q)$. First let $q \in \mathsf{Q}\backslash i(\mathsf{P})$. We consider the case $r \in \mathrm{succ}_{\mathsf{Q}}(q)$, as the other case is similar. If $r \in \mathsf{Q}\backslash i(\mathsf{P})$, then $r \leqslant q$, and they are both in the same chain of $\mathsf{Q}\backslash i(\mathsf{P})$, and hence $\sigma(r) \leqslant \sigma(q)$, since σ is order-preserving on $\mathsf{Q}\backslash i(\mathsf{P})$. If $r \in i(\mathsf{P})$, then $\sigma(r) = i^{-1}(r) \in \mathrm{succ}_{\mathsf{P}}(q)$, and (12) implies $\sigma(r) \in \downarrow_{\mathsf{P}} \sigma(q)$ so that $\sigma(r) \leqslant \sigma(q)$. Now let $q \in i(\mathsf{P})$. Again, we consider the case $r \in \mathrm{succ}_{\mathsf{Q}}(q)$, as the other case is similar. If $r \in \mathsf{Q}\backslash i(\mathsf{P})$, then $\sigma(q) = i^{-1}(q) \in \mathrm{pred}_{\mathsf{P}}(r)$, and (12) implies $\sigma(q) \in \uparrow_{\mathsf{P}} \sigma(r)$ so that $\sigma(r) \leqslant \sigma(q)$. Finally if $r \in i(P)$, then $\sigma(r) \leqslant \sigma(q)$, since $\sigma(r) = i^{-1}(r)$ and $\sigma(q) = i^{-1}(q)$ and i is an order embedding. ∎

Before we state the algorithm for constructing an order retraction, we provide a description of the dictionary data structure. A dictionary D is a set of ordered pairs (k, v) where the *key* k is mapped to its *value* v, and we write $D[k] = v$. In our algorithm, a key is an element $q \in \mathsf{Q}$, and its corresponding value v is a set of elements of P, which are the possible *candidates* in P for $\sigma(q)$, and they are denoted by $\mathsf{cds}[q]$.

Order Retraction Algorithm: The construction of an order retraction σ proceeds in three steps. First we create a dictionary $\mathsf{cds}[q]$ by examining the reachability of elements of P from every q. If $\mathsf{cds}[q] \neq \varnothing$ for each $q \in \mathsf{Q}$, the second step is to repeatedly trim the dictionary values to remove elements that are not compatible with an order-preserving map. Again if $\mathsf{cds}[q] \neq \varnothing$ for each $q \in \mathsf{Q}$, the third step is to one-by-one repeatedly choose a single value $p \in \mathsf{cds}[q]$ for each $q \in \mathsf{Q}$ and trim incompatible values from the dictionary after each choice. This step continues as long as the dictionary values are all nonempty.

Since the sets are finite, this process must eventually halt with each set of dictionary values containing exactly one element. In which case σ is obtained by assigning $\sigma(q) = p$ where $\mathsf{cds}[q] = \{p\}$ for all $q \in \mathsf{Q}$.

Step 1: Initialize cds. For each $p \in \mathsf{P}$, let $\mathsf{cds}[i(p)] = \{p\}$. Now consider $q \in \mathsf{Q}\backslash i(\mathsf{P})$. Algorithmically, we find all $p \in \mathsf{P}$ such that $\mathrm{succ}_{\mathsf{P}}(q) \subset \downarrow_{\mathsf{P}} p$ and $\mathrm{pred}_{\mathsf{P}}(q) \subset \uparrow_{\mathsf{P}} p$ and initialize

$$\mathsf{cds}[q] = \{p \in \mathsf{P} \mid \mathrm{succ}_{\mathsf{P}}(q) \subset \downarrow_{\mathsf{P}} p \text{ and } \mathrm{pred}_{\mathsf{P}}(q) \subset \uparrow_{\mathsf{P}} p\}. \tag{13}$$

Note that $\mathsf{cds}[q]$ could be empty. However, if $p \in \mathsf{cds}[q]$ and $\sigma(q)$ were defined to be p, then condition (12) would be satisfied at q.

Step 2: Trim cds. For each $q \in \mathsf{Q}\backslash i(\mathsf{P})$ we trim $\mathsf{cds}[q]$ as follows. For each $r \in \mathrm{succ}_{\mathsf{Q}}(q)$ remove each element p from $\mathsf{cds}[q]$ for which there is no $t \in \mathsf{cds}[r]$ such that $t \in \downarrow_{\mathsf{P}} p$. Next for each $s \in \mathrm{pred}_{\mathsf{Q}}(q)$ remove each element p from $\mathsf{cds}[q]$ for which there is no $t \in \mathsf{cds}[s]$ such that $t \in \uparrow_{\mathsf{P}} p$. This trimming step is repeated until at least one candidate set is empty or no more candidate sets can be trimmed. If there is $q \in \mathsf{Q}$ for which $\mathsf{cds}[q] = \varnothing$, then an order retraction does not exist and the algorithm terminates at this step.

Step 3: Construct a retraction. Fix $q \in \mathsf{Q}$ with $\#\mathsf{cds}[q] > 1$. Choose an arbitrary element $p \in \mathsf{cds}[q]$ and reassign $\mathsf{cds}[q] = \{p\}$. Then repeat Step 2 above to trim candidate values that are not compatible with an order-preserving map. Now repeatedly make these choices until every candidate set has exactly one element.

2.4 Theorem Let $i : \mathsf{P} \hookrightarrow \mathsf{Q}$ be an order embedding of finite posets. The Order Retraction Algorithm determines the existence or nonexistence of an order retraction $\sigma : \mathsf{Q} \twoheadrightarrow \mathsf{P}$ of i. Moreover, if an order retraction exists, then the algorithm constructs one. ■

Proof. From Step 1, if all $\mathsf{cds}[q]$ are nonempty, then condition (12) in Lemma 2.3 is satisfied. Also note that in Step 2, we do not trim $\mathsf{cds}[i(p)] = \{p\}$ for $p \in \mathsf{P}$, as $\sigma(i(p)) = p$ is required for σ to be an order retraction of i.

Now we prove that if for every $q \in \mathsf{Q}$ we have $\#\mathsf{cds}[q] = 1$ and we define $\sigma : \mathsf{Q} \to \mathsf{P}$ by setting $\sigma(q) = p$ when $\mathsf{cds}[q] = \{p\}$, then σ is an order retraction of i. Since $\mathsf{cds}[i(p)] = \{p\}$ for $p \in \mathsf{P}$, σ is a retraction of i. As noted above, for each q condition (12) in Lemma 2.3 is satisfied. Therefore, by Lemma 2.3 σ is an order retraction if σ is order-preserving on each chain in $\mathsf{Q}\backslash i(\mathsf{P})$. To show this, it is equivalent to show that if $q_1, q_2 \in \mathsf{Q}\backslash i(\mathsf{P})$ with $q_1 \leqslant q_2$, then $\sigma(q_1) \leqslant \sigma(q_2)$. Since $\mathsf{cds}[q_i] = \{\sigma(q_i)\}$, we must have $\sigma(q_2) \in \uparrow_{\mathsf{P}} \sigma(q_1)$, because otherwise, $\sigma(q_2)$ would have been trimmed from $\mathsf{cds}[q_2]$ in the trimming step. Thus $\sigma(q_1) \leqslant \sigma(q_2)$, and hence σ is an order retraction of i.

Claim: If the trimmed candidates sets are all nonempty, then Step 3 above always produces an order retraction. The particular retraction that is constructed depends on the choices that are made in Step 3, but every sequence of choices builds an retraction.

We show by contradiction that at each trim operation in Step 3, the candidates set that is being trimmed does not become empty. Suppose $\mathsf{cds}[q]$ is being trimmed to the empty set because $r \in \mathrm{succ}_{\mathsf{P}}(q)$ for some r with $\mathsf{cds}[r] \neq \varnothing$. An element $p \in \mathsf{cds}[q]$ is removed when no element $t \in \mathsf{cds}[r]$ exists such that $t \in \downarrow_{\mathsf{P}} p$. However, since the trimming at r had been completed with $\mathsf{cds}[r] \neq \varnothing$, for each $t_* \in \mathsf{cds}[r]$ there must exist $p_* \in \mathsf{cds}[q]$ such that $p_* \in \uparrow_{\mathsf{P}} t_*$ so that $t_* \in \downarrow_{\mathsf{P}} p_*$, which contradicts the removal of p_* from $\mathsf{cds}[q]$. An analogous argument holds for the only other possible trim operation where $\mathsf{cds}[q]$ is being trimmed to the empty set because $s \in \mathrm{pred}_{\mathsf{P}}(q)$ for some s with $\mathsf{cds}[s] \neq \varnothing$. This proves the claim.

We have shown that, after the trimming in Step 2, if the candidates dictionary has all nonempty values, there is a procedure given in Step 3 that must construct an order retraction. We must prove the converse, i.e. if an order retraction exists, then the trimmed candidates dictionary must have all nonempty values, so that the above algorithm can establish non-existence of an order retraction by detecting an empty trimmed candidates set.

Claim: If an order retraction exists, then the trimmed candidates dictionary obtained after Step 2 must have all nonempty values.

Suppose $\sigma : \mathsf{Q} \to \mathsf{P}$ is an order retraction of i. Then for each $q \in \mathsf{Q}\backslash i(\mathsf{P})$, we must have $\mathrm{succ}_{\mathsf{P}}(q) \subset \downarrow_P \sigma(q)$ and $\mathrm{pred}_{\mathsf{P}}(q)) \subset \uparrow_P \sigma(q)$, because σ is order-preserving. Therefore (13) implies that $\sigma(q)$ is an element of $\mathsf{cds}[q]$ after Step 1 before trimming.

Moreover, consider $r \in \mathrm{succ}_{\mathsf{P}}(q)$ for some $q \in \mathsf{Q}\backslash i(\mathsf{P})$. Since $\sigma(r) \leqslant \sigma(q)$, because σ is order-preserving, $\sigma(r) \in \downarrow_{\mathsf{P}} \sigma(q)$, and hence $\sigma(q)$ is not trimmed from $\mathsf{cds}[q]$. Likewise, if $s \in \mathrm{pred}_{\mathsf{P}}(q)$, then $\sigma(s) \in \uparrow_{\mathsf{P}} \sigma(q)$, and hence $\sigma(q)$ is not trimmed from $\mathsf{cds}[q]$. Therefore, throughout Step 2, $\sigma(q)$ is never trimmed from $\mathsf{cds}[q]$ for all $q \in \mathsf{Q}\backslash i(\mathsf{P})$. This proves the claim and completes the proof of Theorem 2.4.

Finally we note that the two above claims establish that after trimming in Step 2, if the candidates sets are all nonempty, then

$$\mathsf{cds}[q] = \{p \in \mathsf{P} \mid \exists \text{ order retraction } \sigma \text{ such that } \sigma(q) = p\}.$$

Therefore, the trimmed candidates sets obtained in Step 2 are independent of the order in which the trimming steps are performed. ■

The following corollary lists some situations in which the existence of an order retraction is readily established. The left part of Figure 1 illustrates one of these cases, and the right part of that figure gives an example where the existence is not as straightforward as the cases listed in the corollary.

2.5 Corollary Let $i : \mathsf{P} \hookrightarrow \mathsf{Q}$ be an order embedding. Suppose $\#\mathrm{succ}_{\mathsf{P}}(q) = 1$ for all $q \in \mathsf{Q}\backslash i(\mathsf{P})$, then defining $\sigma(q)$ to be the unique element of $\mathrm{succ}_{\mathsf{P}}(q)$ defines an order retraction of i. Likewise if $\#\mathrm{pred}_{\mathsf{P}}(q) = 1$ for all $q \in \mathsf{Q}\backslash i(\mathsf{P})$, then defining $\sigma(q)$ to be the unique element of $\mathrm{pred}_{\mathsf{P}}(q)$ defines an order retraction of i. Moreover, if either $\#\mathrm{succ}_{\mathsf{P}}(q) = 1$ or $\#\mathrm{pred}_{\mathsf{P}}(q) = 1$ for all $q \in \mathsf{Q}\backslash i(\mathsf{P})$, then an order retraction of i exists with the property that $\sigma(q) \in \mathrm{succ}_{\mathsf{P}}(q) \cup \mathrm{pred}_{\mathsf{P}}(q)$ for all $q \in \mathsf{Q}\backslash i(\mathsf{P})$. ▪

2.6 Remark Let U be the poset of unassignable elements in Q after Step 2. Then $\mathsf{P}^\dagger = \mathsf{P} \cup \mathsf{U}$ with the induced order from Q is a poset with $\mathsf{P} \hookrightarrow \mathsf{P}^\dagger \hookrightarrow \mathsf{Q}$. Applying the algorithm in Theorem 2.4 produces nonempty candidate sets after Step 2, which implies there is an order retraction $\mathsf{Q} \twoheadrightarrow \mathsf{P}^\dagger$. This is the crudest way to find an intermediate poset $\mathsf{P}^\dagger$. If you proceed by augmenting P with elements from U one at a time, applying the algorithm after each step, a smaller intermediate poset $\mathsf{P}^\dagger$ can possibly be constructed along with an order retraction $\mathsf{Q} \twoheadrightarrow \mathsf{P}^\dagger$. ▪

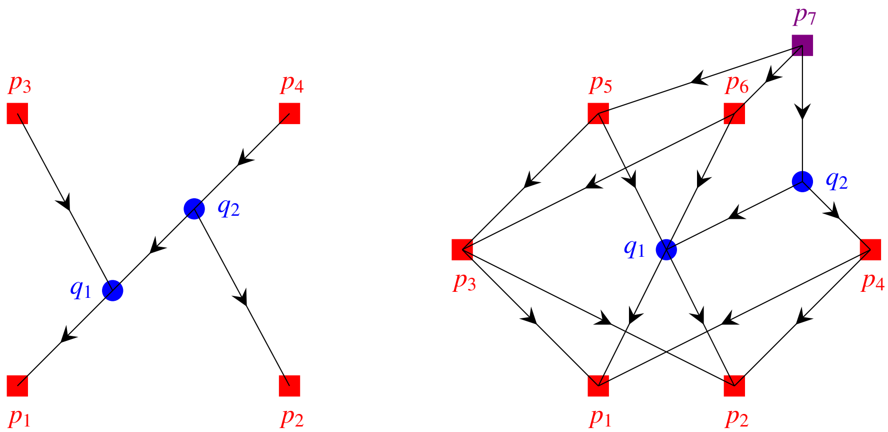


FIGURE 1. Two order embeddings $\mathsf{P} \hookrightarrow \mathsf{Q}$. Left: $\mathrm{succ}_{\mathsf{P}}(q_1) = \{p_1\}$, $\mathrm{pred}_{\mathsf{P}}(q_1) = \{p_3, p_4\}$, $\mathrm{succ}_{\mathsf{P}}(q_2) = \{p_1, p_2\}$, $\mathrm{pred}_{\mathsf{P}}(q_2) = \{p_4\}$, $\mathsf{cds}[q_1] = \{p_1\}$, $\mathsf{cds}[q_2] = \{p_4\}$ and so $\sigma(q_1) = p_1$, $\sigma(q_2) = p_4$ is an order retraction. Right: $\mathrm{succ}_{\mathsf{P}}(q_1) = \{p_1, p_2\}$, $\mathrm{pred}_{\mathsf{P}}(q_1) = \{p_5, p_6\}$, $\mathrm{succ}_{\mathsf{P}}(q_2) = \{p_4\}$, $\mathrm{pred}_{\mathsf{P}}(q_2) = \{p_7\}$. Before trimming $\mathsf{cds}[q_1] = \{p_3\}$, $\mathsf{cds}[q_2] = \{p_4, p_7\}$. Since $p_3 \parallel p_4$, we must trim p_4 from $\mathsf{cds}[q_2]$. Then $\sigma(q_1) = p_3$, $\sigma(q_2) = p_7$ is an order retraction. If p_7 is removed from P, then $\mathrm{pred}_{\mathsf{P}}(q_2) = \varnothing$ and $\mathsf{cds}[q_2] = \{p_4, \}$. Then p_4 is trimmed from $\mathsf{cds}[q_2]$ so that $\mathsf{cds}[q_2] = \varnothing$ and no order retraction exists. Additionally, ignoring the order relation between q_1 and q_2, $\sigma(q_1) = p_3$ and $\sigma(q_2) = p_4$ is an order retraction.

3. **Dynamics of binary relations.**

In this section we consider the global dynamical structure of binary relations. Let $\mathcal{F} \subset \mathcal{X} \times \mathcal{X}$ be a binary relation on $\mathcal{X}$. Then for every $\mathcal{U} \subset \mathcal{X}$ the relation acts on $\mathcal{U}$ as follows:

$$\mathcal{F}(\mathcal{U}) := \{\eta \in \mathcal{X} \mid \exists\, \xi \in \mathcal{U} \text{ such that } (\xi, \eta) \in \mathcal{F}\}.$$

This defines a map on $\mathsf{Set}(\mathcal{X})$ whose elementwise representation is a multivalued map on $\mathcal{X}$, cf. [19, 20]. The inverse map $\mathcal{F}^{-1}$ is obtained by considering the *opposite* relation in which the order of the pairs is reversed. The concept of binary relation can be equivalently described by the notion of *directed graph* as follows: the set $\mathcal{X}$ represent the vertices and the edges are given by the pairs $(\xi, \eta) \in \mathcal{F}$, where ξ is the source and η the target. In terms of the corresponding directed graphs, $\mathcal{F}^{-1}$ has the same vertices and edges as $\mathcal{F}$ but with the direction of the edges reversed. We abuse notation and use the symbol $\mathcal{F}$ to represent both a binary relation on $\mathcal{X}$ and its equivalent digraph.

3.1. **Fundamental lattice structures.**

Recall from Section 1.2 that a set $\mathcal{U} \subset \mathcal{X}$ is *forward invariant* if $\mathcal{F}(\mathcal{U}) \subset \mathcal{U}$. A set $\mathcal{A} \subset \mathcal{X}$ is an *attractor* for $\mathcal{F}$ if $\mathcal{F}(\mathcal{A}) = \mathcal{A}$. Forward invariant sets and attractors are denoted by $\mathsf{Invset}^+(\mathcal{F})$ and $\mathsf{Att}(\mathcal{F})$ respectively. By [20, Prop. 2.3] the set $\mathsf{Invset}^+(\mathcal{F})$ is a finite distributive lattices with respect to intersection and union.

From [16, 20] also recall the definition of *ω-limit set* of a set $\mathcal{U} \subset \mathcal{X}$

$$\omega(\mathcal{U}, \mathcal{F}) = \bigcap_{k \geqslant 0} \Gamma_k^+(\mathcal{U}) \tag{14}$$

where $\Gamma_k^+(\mathcal{U}) = \bigcup_{n \geqslant k} \mathcal{F}^n(\mathcal{U})$ for $k > 0$ is *k-forward image* of $\mathcal{U}$ and $\Gamma^+(\mathcal{U}) = \Gamma_0^+(\mathcal{U})$. If there is no ambiguity about $\mathcal{F}$ we write $\omega(\cdot)$ instead of $\omega(\cdot, \mathcal{F})$. For attractors $\mathcal{A}, \mathcal{A}' \in \mathsf{Att}(\mathcal{F})$ define

$$\mathcal{A} \vee \mathcal{A}' = \mathcal{A} \cup \mathcal{A}' \quad \text{and} \quad \mathcal{A} \wedge \mathcal{A}' = \omega(\mathcal{A} \cap \mathcal{A}').$$

By [20, Prop. 2.5], the set $\mathsf{Att}(\mathcal{F})$, with $\wedge$ and $\vee$ as defined above, is a finite distributive lattice, and by [20, Prop. 2.7] the map

$$\omega \colon \mathsf{Invset}^+(\mathcal{F}) \twoheadrightarrow \mathsf{Att}(\mathcal{F}) \tag{15}$$

is a lattice epimorphism.

3.2. **Recurrence and strong connectivity.** In a finite directed graph, recurrent behavior is characterized by the vertices that lie on a cycle. In terms of the corresponding binary relation $\mathcal{F}$, an element ξ lies on a cycle if and only if it is periodic, i.e. there exists $n > 0$ such that $\xi \in \mathcal{F}^n(\xi)$. In the graph-theoretic context, we describe cyclic vertices in terms of reachability and connectivity. If there exists a sequence $\{\xi_0, \cdots, \xi_k\}$, with $\xi_0 = \xi$ and $\xi_k = \xi'$, such that $\xi_{i+1} \in \mathcal{F}(\xi_i)$, then ξ' is *reachable* from ξ, denoted by $\xi' \leftarrow \xi$. The reachability relation is the transitive closure $\mathcal{F}^+$ of the relation $\mathcal{F}$. We use the convention that $\xi' \leftarrow \xi$ is equivalent to $\xi \rightarrow \xi'$. If $\xi \leftarrow \xi'$ and $\xi' \leftarrow \xi$, then ξ is *connected* to ξ', denoted by $\xi \leftrightarrow \xi'$. If $\xi \not\leftarrow \xi'$ and $\xi' \not\leftarrow \xi$, then ξ, ξ' are *parallel* elements, denoted by $\xi \parallel \xi'$. The relation $\leftrightarrow$ is symmetric and transitive and hence defines the *partial* equivalence relation of *connectivity* on $\mathcal{X}$. It is an equivalence relation on the set of cyclic vertices, and the partial equivalence classes of $\leftrightarrow$ are called the *cyclic strongly connected components* $[\xi]_{\leftrightarrow}$ of $\mathcal{F}$. The latter terminology is used in graph theory, but here we refer to them as the *recurrent components* which reflects dynamical systems terminology. We denote the set of recurrent components by $\mathsf{RC}(\mathcal{F})$. The following lemma is a direct consequence of Proposition 3.4 in [16].

3.1 Lemma If $\mathcal{S} \in \mathsf{RC}(\mathcal{F})$, then $\mathcal{S}$ is invariant, i.e. $\mathcal{S} \subset \mathcal{F}(\mathcal{S})$ and $\mathcal{S} \subset \mathcal{F}^{-1}(\mathcal{S})$. ∎

Via the reachability relation we define a partial order on $\mathsf{RC}(\mathcal{F})$ as follows. For $\mathcal{S}, \mathcal{S}' \in \mathsf{RC}(\mathcal{F})$, we say $\mathcal{S}' \leqslant \mathcal{S}$ if and only if there exist $\xi \in \mathcal{S}$ and $\xi' \in \mathcal{S}'$, such that $\xi' \leftarrow \xi$. Anti-symmetry and transitivity follow from $\leftrightarrow$, and reflexivity follows since the sets $\mathcal{S} \in \mathsf{RC}(\mathcal{F})$ are invariant. We refer to the poset $\big(\mathsf{RC}(\mathcal{F}), \leqslant\big)$ as the poset of *recurrent components* of $\mathcal{F}$, which is also referred to as the Morse graph in [1].

To make $\leftrightarrow$ into an equivalence relation on the set of all vertices $\mathcal{X}$, we consider its reflexive closure $\circlearrowleft = (\leftrightarrow)^=$, i.e. given $\xi, \xi' \in \mathcal{X}$ define $\xi \circlearrowleft \xi'$ if $\xi \leftrightarrow \xi'$ or $\xi' = \xi$. The equivalence relation $\circlearrowleft$ on $\mathcal{X}$ is called *strong connectivity*. In graph theory the equivalence classes $[\xi]_{\circlearrowleft}$ are called the *strongly connected components* which we denote by $\mathsf{SC}(\mathcal{F})$. Note that unlike the recurrent components, the elements of $\mathsf{SC}(\mathcal{F})$ are not necessarily invariant sets for $\mathcal{F}$. A singleton set containing a non-cyclic vertex is a strongly connected component but is not invariant, and the following lemma implies that this is the only such example.

3.2 Lemma If $\mathcal{S} \in \mathsf{SC}(\mathcal{F})$ and $\#\mathcal{S} > 1$, then $\mathcal{S}$ is invariant. ∎

Proof. Denote the invariant sets of $\mathcal{F}$ by $\mathsf{Invset}(\mathcal{F})$. By definition $\mathsf{RC}(\mathcal{F}) \subset \mathsf{SC}(\mathcal{F})$ so by Lemma 3.1 we have $\mathsf{RC}(\mathcal{F}) \subset \mathsf{SC}(\mathcal{F}) \cap \mathsf{Invset}(\mathcal{F})$. If $\mathcal{S} \in \mathsf{SC}(\mathcal{F})$ and $\#\mathcal{S} > 1$, then for every $\xi \in \mathcal{S}$ we have $\xi' \in \mathcal{S}$ with $\xi' \leftarrow \xi$ and $\xi \leftarrow \xi'$ so that $\xi \leftrightarrow \xi'$, which implies $\mathcal{S} \in \mathsf{RC}(\mathcal{F}) \subset \mathsf{Invset}(\mathcal{F})$. Therefore, if $\mathcal{S} \in \mathsf{SC}(\mathcal{F}) \backslash \mathsf{RC}(\mathcal{F})$, then $\mathcal{S}$ contains a singleton non-cyclic vertex ξ, which implies that $\mathcal{S} = \{\xi\} \not\subset \mathcal{F}(\{\xi\}) = \mathcal{F}(\mathcal{S})$ so that $\mathcal{S} \notin \mathsf{Invset}(\mathcal{F})$. ■

The partial order on $\mathsf{RC}(\mathcal{F})$ can be extended to a partial order on $\mathsf{SC}(\mathcal{F})$ as follows. Given $\mathcal{S}, \mathcal{S}' \in \mathsf{SC}(\mathcal{F})$ define $\mathcal{S}' \leqslant \mathcal{S}$ if there exist $\xi \in \mathcal{S}$ and $\xi' \in \mathcal{S}'$ such that $\xi' \leftarrow \xi$ or $\mathcal{S} = \mathcal{S}'$. We refer to the poset $\big(\mathsf{SC}(\mathcal{F}), \leqslant\big)$ as the poset of *strongly connected components* of $\mathcal{F}$. Set inclusion defines the order embedding

$$\iota \colon \mathsf{RC}(\mathcal{F}) \hookrightarrow \mathsf{SC}(\mathcal{F}). \tag{16}$$

4. **The Birkhoff representation theorem for binary relations.** In this section we present a generalization of Birkhoff's representation theorem for finite binary relations. We start off with the classical Birkhoff representation theorem.

4.1 Theorem (Birkhoff's representation theorem [7]) Let L be a finite distributive lattice and let P be a finite poset. Then, $\lambda \colon \mathsf{L} \to \mathsf{O}(\mathsf{J}(\mathsf{L}))$, defined by $a \mapsto \{a' \in \mathsf{J}(\mathsf{L}) \mid a' \leqslant a\}$, is a lattice isomorphism, and $\mu \colon \mathsf{P} \to \mathsf{J}(\mathsf{O}(\mathsf{P}))$, defined by $p \mapsto \downarrow p$, is an order isomorphism. Morover, the mappings $a \mapsto \bigvee \lambda(a)$ and $p \mapsto \sup \mu(p)$ are the identity mappings on L and P respectively. ■

With respect to morphisms in the categories **FDLat** and **FPoset** we have the following construction.

4.2 Theorem (Theorem 5.19 in [7], Theorem 10.4 in [24]) (i) Let K, L be finite distributive lattices. Given a lattice homomorphism $h : \mathsf{K} \to \mathsf{L}$, define an associated order-preserving map $\phi_h : \mathsf{J}(\mathsf{L}) \to \mathsf{J}(\mathsf{K})$ given by

$$\phi_h(a) = \min h^{-1}(\uparrow a) \quad \text{for} \quad a \in \mathsf{J}(\mathsf{L})$$

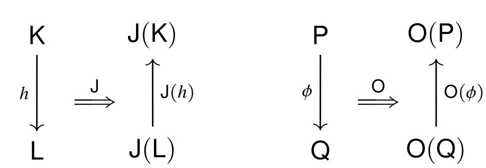
where the up-arrow indicates the up-set $\uparrow a := \{q \mid q \geqslant a\}$ in L. A homomorphism h is a lattice monomorphism if and only if ϕ_h is a order surjection and h is a lattice epimorphism if and only if ϕ_h is an order embedding.

(ii) Let P, Q be finite posets. Given an order preserving map $\phi : \mathsf{P} \to \mathsf{Q}$, define an associated lattice homomorphism $h_\phi : \mathsf{O}(\mathsf{Q}) \to \mathsf{O}(\mathsf{P})$ given by

$$h_\phi(I) = \phi^{-1}(I), \quad I \in \mathsf{O}(\mathsf{Q}).$$

An order-preserving map ϕ is an order embedding if and only h_ϕ is a lattice epimorphism and ϕ is an order surjection if and only if h_ϕ is a lattice monomorphism. ■

If we define $\mathsf{J}(h) = \phi_h$ and $\mathsf{O}(\phi) = h_\phi$, then J and O define contrvariant functors which are referred to as the *join* and *down-set* functor respectively:



If we use the above construction in combination with Birkhoff's representation theorem then for $h\colon \mathsf{O}(\mathsf{Q}) \to \mathsf{O}(\mathsf{P})$, the map $\mathsf{J}(h)$ induces a map $\phi\colon \mathsf{P} \to \mathsf{Q}$ given by $\phi(p) = \min\{q \in \mathsf{Q} \mid p \in h(\downarrow q)\}$.

4.1. **Join-irreducible attractors.** The join-irreducible elements of the lattice $\mathsf{Att}(\mathcal{F})$ can be characterized as follows.

4.3 Lemma $\mathsf{J}\big(\mathsf{Att}(\mathcal{F})\big) = \{\Gamma^+(\xi) \mid [\xi]_{\leftrightarrow} \in \mathsf{RC}(\mathcal{F})\}$. ▪

Proof. We first show that every $\Gamma^+(\xi)$ is join-irreducible. Suppose $\Gamma^+(\xi) = \mathcal{A} \cup \mathcal{A}'$ for some $\mathcal{A}, \mathcal{A}' \in \mathsf{Att}(\mathcal{F})$. Assume, without loss of generality, that $\xi \in \mathcal{A}$, then $\Gamma^+(\xi) \subset \mathcal{A}$, since $\mathcal{A} \in \mathsf{Att}(\mathcal{F})$. By assumption $\mathcal{A} \subset \Gamma^+(\xi)$ so that $\Gamma^+(\xi) = \mathcal{A}$, which proves that $\Gamma^+(\xi)$ is join-irreducible.

Let $\mathcal{A} \in \mathsf{Att}(\mathcal{F})$, then $\mathcal{A} = \bigcup_{\xi\in\mathcal{A}} \Gamma^+(\xi)$. Note that $\xi' \leftarrow \xi$ if and only if $\Gamma^+(\xi') \subset \Gamma^+(\xi)$. Therefore if $\xi \parallel \xi'$, then $\Gamma^+(\xi') \parallel \Gamma^+(\xi')$ in $\mathsf{Att}(\mathcal{F})$. This implies that in the above union we only need parallel elements ξ, i.e. $\mathcal{A}$ is a union $\Gamma^+(\xi_i)$ for finitely many $\xi_i \in \mathcal{A}$, with $\xi_i \parallel \xi_j$ and $i \neq j$, which is an irredundant, join-irreducible representation of $\mathcal{A}$. This proves that join-irreducible elements are of the form $\Gamma^+(\xi)$. Since $\mathcal{F}(\mathcal{A}) = \mathcal{A} = \Gamma^+(\xi)$, there exists $\eta \in \mathcal{A}$ such that $\xi \in \mathcal{F}(\eta)$, which implies $\xi \rightarrow \eta \rightarrow \xi$ so that $\xi \leftrightarrow \xi$, i.e. $[\xi]_{\leftrightarrow} \in \mathsf{RC}(\mathcal{F})$. ∎

4.4 Lemma The map $\Gamma^+\colon \mathsf{RC}(\mathcal{F}) \to \mathsf{J}\big(\mathsf{Att}(\mathcal{F})\big)$ defined by

$$[\xi]_{\leftrightarrow} \mapsto \Gamma^+(\xi), \tag{17}$$

is an order isomorphism. ▪

Proof. By definition, $[\xi]_{\leftrightarrow} \leqslant [\eta]_{\leftrightarrow}$ if and only if $\Gamma^+(\xi) \subset \Gamma^+(\eta)$. Moreover, $\Gamma^+(\xi) = \Gamma^+(\eta)$ implies $\xi \rightarrow \eta$ and $\eta \rightarrow \xi$ so that $\xi \circlearrowleft \eta$. Consequently $[\xi]_{\leftrightarrow} = [\eta]_{\leftrightarrow}$, and the map is injective. Surjectivity follows from Lemma 4.3. ∎

In Appendix A we show that $\mathsf{Invset}^+(\mathcal{F})$ is the attractor lattice of the reflexive closure relation $\mathcal{F}^=$, i.e. $\mathsf{Invset}^+(\mathcal{F}) = \mathsf{Att}(\mathcal{F}^=)$ and $\mathsf{SC}(\mathcal{F}) = \mathsf{RC}(\mathcal{F}^=)$ cf. Lemma A.5. If we combine Lemma 4.4 with Lemma A.5 we obtain $\mathsf{J}\big(\mathsf{Invset}^+(\mathcal{F})\big) = \{\Gamma^+(\xi) \mid [\xi]_{\circlearrowleft} \in \mathsf{SC}(\mathcal{F})\} = \{\Gamma^+(\xi) \mid \xi \in \mathcal{X}\}$. From the contravariance of the join functor we obtain the following commutative diagram:

$$\begin{array}{ccc}
\mathsf{SC}(\mathcal{F}) & \xleftrightarrow[\cong]{\Gamma^+} & \mathsf{J}\big(\mathsf{Invset}^+(\mathcal{F})\big) \\
{\scriptstyle \subset}\big\uparrow{\scriptstyle \iota} & & \big\uparrow{\scriptstyle \mathsf{J}(\omega)} \\
\mathsf{RC}(\mathcal{F}) & \xleftrightarrow[\cong]{\Gamma^+} & \mathsf{J}\big(\mathsf{Att}(\mathcal{F})\big)
\end{array} \tag{18}$$

Let $\xi \in [\xi]_{\leftrightarrow}$, then $\xi \mapsto \Gamma^+(\xi) \in \mathsf{J}(\mathsf{Att}(\mathcal{F}))$ by Lemma 4.4. Since $\omega|_{\mathsf{Att}(\mathcal{F})} = \mathrm{id}$ and $\mathsf{Att}(\mathcal{F}) \subset \mathsf{Invset}^+(\mathcal{F})$, we have $\mathsf{J}(\omega)(\Gamma^+(\xi)) = \min \omega^{-1}(\uparrow \Gamma^+(\xi)) = \Gamma^+(\xi)$ using Theorem 4.2, which proves that $\iota\colon \mathsf{RC}(\mathcal{F}) \hookrightarrow \mathsf{SC}(\mathcal{F})$ in Diagram (18) is the inclusion map.

4.5 Remark In [21] we introduced the Conley form on bounded distributive lattices. The dual of an attractor $\mathcal{A}$ of $\mathcal{F}$ is the maximal subset $\mathcal{A}^*$ of $\mathcal{A}^c$ such that $\mathcal{F}^{-1}(\mathcal{A}^*) = \mathcal{A}^*$. Via the Conley form, $\mathsf{J}(\mathsf{Att}(\mathcal{F}))$ may be represented as $\mathsf{J}(\mathsf{Att}(\mathcal{F})) \cong \{\mathcal{A} \cap \overleftarrow{\mathcal{A}}^* \mid \mathcal{A} \in \mathsf{J}(\mathsf{Att}(\mathcal{F}))\}$. Since $\mathcal{A} \cap \overleftarrow{\mathcal{A}}^*$ is in $\mathsf{RC}(\mathcal{F})$, we obtain that the inverse of $\Gamma^+\colon \mathsf{RC}(\mathcal{F}) \to$

$\mathsf{J}(\mathsf{Att}(\mathcal{F}))$ is given by $\mathcal{A} \mapsto \mathcal{A} \cap \overleftarrow{\mathcal{A}}^{*}$. If we apply the Conley form to $\mathcal{F}^{=}$, then the inverse of $\Gamma^{+}\colon \mathsf{SC}(\mathcal{F}) \to \mathsf{J}(\mathsf{Invset}^{+}(\mathcal{F}))$ is given by $\mathcal{U} \mapsto \mathcal{U} \cap \overleftarrow{\mathcal{U}}^{c}$. ▪

4.2. **Proofs of Theorems 1.5 and 1.6.** The commutative diagram in (18) provides a lattice epimorphism in Theorem 1.6 given by $\omega\colon \mathsf{Invset}^{+}(\mathcal{F}) \twoheadrightarrow \mathsf{Att}(\mathcal{F})$. If $h\colon \mathsf{K} \twoheadrightarrow \mathsf{L}$ is another lattice epimorphism satisfying Diagram (9) then via the down-set functor we obtain the commuting diagram in (8), which shows that the lattice epimorphism in Theorem 1.6 is unique up to isomorphism and completes the proof of Theorem 1.6.

If we apply the down-set functor to Diagram (18) we obtain the following commutative diagram

$$\begin{array}{ccccc} \mathsf{O}\big(\mathsf{SC}(\mathcal{F})\big) & \xleftrightarrow{\cong} & \mathsf{O}\big(\mathsf{J}(\mathsf{Invset}^{+}(\mathcal{F}))\big) & \xleftrightarrow{\cong} & \mathsf{Invset}^{+}(\mathcal{F}) \\ \Big\downarrow{\scriptstyle \mathsf{O}(\iota)} & & {\scriptstyle \mathsf{O}(\mathsf{J}(\omega))}\Big\downarrow & & {\scriptstyle \omega}\Big\downarrow \\ \mathsf{O}\big(\mathsf{RC}(\mathcal{F})\big) & \xleftrightarrow{\cong} & \mathsf{O}\big(\mathsf{J}(\mathsf{Att}(\mathcal{F}))\big) & \xleftrightarrow{\cong} & \mathsf{Att}(\mathcal{F}) \end{array} \tag{19}$$

The maps in the above diagram can be determined as follows.

4.6 Lemma The isomorphism $\jmath'\colon \mathsf{O}\big(\mathsf{RC}(\mathcal{F})\big) \to \mathsf{Att}(\mathcal{F})$ is given by

$$I \mapsto \bigcup_{[\xi]_{\nLeftrightarrow} \in I} \Gamma^{+}(\xi).$$

▪

Proof. The map $\mathsf{O}(\mathsf{RC}(\mathcal{F})) \to \mathsf{O}\big(\mathsf{J}(\mathsf{Att}(\mathcal{F}))\big)$ is given by $I \mapsto \Gamma^{+}(I)$ by Lemma 4.4. By Theorem 4.1 the composition

$$I \mapsto \{\Gamma^{+}(\xi) \mid [\xi]_{\nLeftrightarrow} \in I\} \mapsto \bigcup_{[\xi]_{\nLeftrightarrow} \in I} \Gamma^{+}(\xi).$$

is the desired isomorphism. ■

The same argument applies to $\mathsf{SC}(\mathcal{F})$.

4.7 Lemma The isomorphism $\jmath\colon \mathsf{O}(\mathsf{SC}(\mathcal{F})) \to \mathsf{Invset}^{+}(\mathcal{F})$ is given by

$$I \mapsto \bigcup_{[\xi]_{\cup} \in I} \Gamma^{+}(\xi).$$

▪

To prove Theorem 1.5 we argue as follows. Given a lattice epimorphism $h\colon \mathsf{K} \twoheadrightarrow \mathsf{L}$, then on the point set $\mathcal{X} = \mathsf{J}(\mathsf{K})$ we define a relation $\mathcal{F}$ as follows: $(\xi,\eta) \in \mathcal{F}$ if $\xi < \eta$ in $\mathsf{J}(\mathsf{K})$ and $(\xi,\xi) \in \mathcal{F}$ if $\xi = \mathsf{J}(h)(\eta)$ for some $\eta \in \mathsf{J}(\mathsf{L})$. Observe that $\mathsf{SC}(\mathcal{F}) = \mathsf{J}(\mathsf{K})$ and $\mathsf{RC}(\mathcal{F}) = \{\mathsf{J}(h)(\eta) \mid \eta \in \mathsf{J}(\mathsf{L})\}$, and we have the commutative diagram

$$\begin{array}{ccc} \mathsf{SC}(\mathcal{F}) & \xleftrightarrow[=]{\mathrm{id}} & \mathsf{J}(\mathsf{K}) \\ {\scriptstyle \iota}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathsf{J}(h)} \\ \mathsf{RC}(\mathcal{F}) & \xleftrightarrow[\cong]{\zeta} & \mathsf{J}(\mathsf{L}) \end{array} \tag{20}$$

where $\zeta\colon \mathsf{J}(\mathsf{L}) \longleftrightarrow \mathsf{RC}(\mathcal{F})$ is given by $\eta \mapsto \min h^{-1}(\eta)$ and $\iota = \mathsf{J}(h) \circ \zeta^{-1}$ is the inclusion map. If we apply the down-set functor to Diagram (20), and combine the latter with Diagram (19), then we obtain the commutative diagram in (8), which proves the existence of a binary relation in Theorem 1.5. Assume that $\mathcal{F}'$ is another binary relation satisfying (8), then by applying the join functor and using Diagram (18), we obtain the commutative diagram in (10), which proves that $\mathcal{F}$ is uniquely determined up to equivalence. This completes the proof of Theorem 1.5.

5. **Combinatorial models for dynamics.** Returning to a combinatorial model $h\colon \mathsf{K} \twoheadrightarrow \mathsf{L}$ in Diagram (5), we can use Theorem 1.5 to represent h by $\omega\colon \mathsf{Invset}^+(\mathcal{F}) \twoheadrightarrow \mathsf{Att}(\mathcal{F})$ for some finite binary relation $(\mathcal{X}, \mathcal{F})$. Moreover, let $\mathsf{N} = e(\mathsf{K})$ and $\mathsf{A} = \omega(\mathsf{N})$. Then we obtain the commutative diagram

$$\begin{array}{ccccccc} \mathsf{K} & \longleftrightarrow & \mathsf{Invset}^+(\mathcal{F}) & \overset{e}{\longleftrightarrow} & \mathsf{N} & \overset{\subset}{\rightarrowtail} & \mathsf{ABlock}_{\mathscr{R}}(f) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} \\ \mathsf{L} & \longleftrightarrow & \mathsf{Att}(\mathcal{F}) & \overset{c}{\dashrightarrow} & \mathsf{A} & \overset{\subset}{\rightarrowtail} & \mathsf{Att}(f) \end{array} \tag{21}$$

The homomorphism $c\colon \mathsf{Att}(\mathcal{F}) \twoheadrightarrow \mathsf{A}$, called the *connecting homomorphism*, is indicated with a dashed arrow because it may not exist for a given combinatorial model. However, this homomorphism is essential for the epimorphism $h\colon \mathsf{K} \to \mathsf{L}$ to reflect the dynamics of $\omega\colon \mathsf{N} \to \mathsf{A}$. Combinatorial models for which the connecting homomorphism exists are called *commutative* combinatorial models. Recall from Section 1.2 that if $\mathcal{F}$ is an outer approximation, then the connecting homomorphism exists as in Diagram (7).

Focusing on the middle square in (21) and dualizing, we obtain the commutative diagrams

$$\begin{array}{ccc} \mathsf{Invset}^+(\mathcal{F}) & \overset{e}{\longleftrightarrow} & \mathsf{N} \\ \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} \\ \mathsf{Att}(\mathcal{F}) & \overset{c}{\dashrightarrow} & \mathsf{A} \end{array} \qquad\qquad \begin{array}{ccc} \mathsf{SC}(\mathcal{F}) & \overset{\delta}{\longleftrightarrow} & \mathsf{T}(\mathsf{N}) \\ {\scriptstyle i}\uparrow{\scriptstyle \subset} & & \uparrow{\scriptstyle \pi} \\ \mathsf{RC}(\mathcal{F}) & \dashleftarrow & \mathsf{M}(\mathsf{A}) \end{array} \tag{22}$$

The diagram on the right is obtained via Diagram (20) and the representations of the Conley form in Equation (4). As described in the introduction, the Conley form represents the join irreducible elements of a lattice as elements in a meet semilattice. In particular, the Conley form for an attractor lattice A is represented in the meet semilattice of invariant sets by $\mathsf{M}\colon A \mapsto A \cap \overleftarrow{A}^{*}$ for $A \in \mathsf{J}(\mathsf{A})$, which induces an order isomorphism $\mathsf{J}(\mathsf{A}) \cong \mathsf{M}(\mathsf{A})$. Moreover, the Conley form for a lattice of regular closed attracting blocks N is represented in the meet semilattice of regular closed sets by $\mathsf{T}\colon N \mapsto N \wedge \overleftarrow{N}^{\#}$ for $N \in \mathsf{J}(\mathsf{N})$, which induces an order isomorphism $\mathsf{J}(\mathsf{N}) \cong \mathsf{T}(\mathsf{N})$. By duality we have $\mathsf{J}(\omega)\colon \mathsf{J}(\mathsf{A}) \hookrightarrow \mathsf{J}(\mathsf{N})$, which is given by $\mathsf{J}(\omega)(A) = \min \omega^{-1}(\uparrow A)$, so that the embedding $\pi\colon \mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N})$ is defined by $\mathsf{M}(A) \mapsto A \mapsto \mathsf{J}(\omega)(A) \mapsto \mathsf{T}(\mathsf{J}(\omega)(A))$ for $A \in \mathsf{A}$.

By Remark 4.5, the Conley form for $\mathsf{Invset}^+(\mathcal{F})$ is represented in the meet semilattice $\mathsf{Set}(\mathcal{X})$ by $\mathcal{U} \mapsto \mathcal{U} \cap \overleftarrow{\mathcal{U}}^{c}$ for $\mathcal{U} \in \mathsf{J}(\mathsf{Invset}^+(\mathcal{F}))$, which induces an order isomorphism $\mathsf{J}(\mathsf{Invset}^+(\mathcal{F})) \cong \mathsf{SC}(\mathcal{F})$, whose inverse is $\mathcal{S} \mapsto \Gamma^+(\mathcal{S})$. By duality we have $\mathsf{J}(e)\colon \mathsf{J}(\mathsf{N}) \hookrightarrow \mathsf{J}(\mathsf{Invset}^+(\mathcal{F}))$, which is given by $\mathsf{J}(e)(N) = \min e^{-1}(\uparrow N)$. Therefore, the isomorphism $\mathsf{SC}(\mathcal{F}) \cong \mathsf{T}(\mathsf{N})$ is given by $\mathcal{S} \mapsto \Gamma^+(\mathcal{S}) \mapsto \mathsf{J}(e)^{-1}(\Gamma^+(\mathcal{S})) \mapsto \mathsf{T}\big(\mathsf{J}(e)^{-1}(\Gamma^+(\mathcal{S}))\big)$ where the latter map is the Conley form on N as above. Finally, as in Remark 4.5, the Coney form for $\mathsf{Att}(\mathcal{F})$ is represented in the meet semilattice $\mathsf{Set}(\mathcal{X})$ by $\mathcal{A} \mapsto \mathcal{A} \cap \overleftarrow{\mathcal{A}}^*$ for $\mathcal{U} \in \mathsf{J}(\mathsf{Att}(\mathcal{F}))$, which induces an order isomorphism $\mathsf{J}(\mathsf{Att}(\mathcal{F})) \cong \mathsf{RC}(\mathcal{F})$.

The information in N and the homomorphism $\omega\colon \mathsf{Invset}^+(\mathcal{F}) \twoheadrightarrow \mathsf{Att}(\mathcal{F})$ are given as part of a combinatorial model. The information in A is not known in general and the dashed arrows may not exist. For outer approximations the dashed arrows in Diagram (7) are determined because an outer approximation cannot mask any of the recurrent behavior of the underlying system so that there must be a map $\mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{RC}(\mathcal{F})$ which induces $\mathsf{Att}(\mathcal{F}) \twoheadrightarrow \mathsf{A}$. In general, without a connecting homomorphism the structure of $\mathsf{Att}(\mathcal{F})$ need not be related at all to the structure of A. Thus, two central questions arise.

(i) In a general $\omega\colon \mathsf{Invset}^+(\mathcal{F}) \twoheadrightarrow \mathsf{Att}(\mathcal{F})$ is not a commutative combinatorial model, and the only Morse decomposition that follows is $\mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N})$. Is there a way to determine whether a connecting homomorphism exists, and if so construct one?

(ii) Even for a commutative combinatorial model, in applications the size of the poset $\mathsf{SC}(\mathcal{F}) \cong \mathsf{T}(\mathsf{N})$ is typically large, while the poset $\mathsf{RC}(\mathcal{F})$ is relatively small, and consequently the induced tesselated Morse decomposition $\mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N})$ may be impractical to use due to the size of $\mathsf{T}(\mathsf{N})$. Can a tesselated Morse decomposition be built with the tesselation isomorphic to $\mathsf{RC}(\mathcal{F})$, i.e. does there exists an order-coarsening of $\mathsf{T}(\mathsf{N})$ which isomorphic to $\mathsf{RC}(\mathcal{F})$?

5.1. **Construction of commutative combinatorial models.** In both of the above questions, the only tesselated Morse decomposition is given by $\mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N})$, and both problems can be addressed with one procedure using the existence and construction of an order-retraction as in Section 2. Let us consider the diagrams in (21) without prior knowledge of the dashed arrows:

$$
\begin{array}{ccc}
\mathsf{Invset}^+(\mathcal{F}) & \xleftrightarrow{\ e\ } & \mathsf{N} \\
\Big\downarrow \omega & & \Big\downarrow \omega \\
\mathsf{Att}(\mathcal{F}) & & \mathsf{A}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathsf{SC}(\mathcal{F}) & \xleftrightarrow{\ \delta\ } & \mathsf{T}(\mathsf{N}) \\
i \Big\uparrow \subset & & \Big\uparrow \pi \\
\mathsf{RC}(\mathcal{F}) & & \mathsf{M}(\mathsf{A})
\end{array}
$$

If an order retraction $\sigma\colon \mathsf{SC}(\mathcal{F}) \twoheadrightarrow \mathsf{RC}(\mathcal{F})$ exists, then the dual map $\ell\colon \mathsf{Att}(\mathcal{F}) \hookrightarrow \mathsf{Invset}^+(\mathcal{F})$ is a lift. Denoting its image by $\mathsf{K}_\ell = \ell(\mathsf{Att}(\mathcal{F}))$ and its realization by $\mathsf{N}_\ell = e(\ell(\mathsf{Att}(\mathcal{F}))$, we have the commutative diagrams

$$
\begin{array}{ccc}
\mathsf{K}_\ell & \xleftrightarrow{\ e\ } & \mathsf{N}_\ell \\
\ell \Big\updownarrow \omega=\ell^{-1} & & \Big\downarrow \omega \\
\mathsf{Att}(\mathcal{F}) & \xrightarrow{\ c\ } & \mathsf{A}_\ell
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathsf{J}(\mathsf{K}_\ell) & \longleftrightarrow & \mathsf{T}(\mathsf{N}_\ell) \\
\Big\updownarrow & & \Big\uparrow \\
\mathsf{RC}(\mathcal{F}) & \longleftarrow & \mathsf{M}(\mathsf{A}_\ell)
\end{array}
$$

where the connecting homomorphism $c = \omega \circ e \circ \ell$ is surjective and induces an order embedding

$$
\mathsf{M}(\mathsf{A}_\ell) \hookrightarrow \mathsf{RC}(\mathcal{F}) \leftrightarrow \mathsf{T}(\mathsf{N}_\ell) \tag{23}
$$

as a tesselated Morse decomposition.

Hence the existence of an order retraction, which can be determined and constructed algorithmically, implies the existence of a connecting homomorphism. However, in general, the lattice of attractors A_ℓ in the resulting commutative combinatorial model can be coarser than A, because some recurrent dynamics of the underlying system may not be covered by $|\mathsf{RC}(\mathcal{F})|$, and different order retractions can result in different attractor lattices A_ℓ, see Example 6.1. As stated in the introduction, the sublattice A is typically not known a priori, and hence coarsening A to A_ℓ (which is also not known precisely) does not cause any problems in practice. This addresses question (i) in the previous section.

Note that the existence of an order retraction also addresses question (ii), because we obtain a tesselated Morse decomposition to $\mathsf{T}(\mathsf{N}_\ell)$ which is isomorphic to $\mathsf{RC}(\mathcal{F})$, and hence typically much smaller than $\mathsf{T}(\mathsf{N})$. If a connecting homomorphism $\widehat{c}\colon \mathsf{Att}(\mathcal{F}) \to \mathsf{A}$ exists a priori, then we have the following lemma.

5.1 Lemma If a connecting homomorphism $\widehat{c}\colon \mathsf{Att}(\mathcal{F}) \twoheadrightarrow \mathsf{A}$ in Diagram (21) exists, then $\widehat{c} = c := \omega \circ e \circ \ell$ and $\mathsf{A}_\ell = \mathsf{A}$. ▪

Proof. Since σ is an order retraction, $\sigma \circ i = \mathrm{id}_{\mathsf{RC}(\mathcal{F})}$ and therefore, by contravariance of O functor, $\omega \circ \ell = \mathrm{id}_{\mathsf{Att}(\mathcal{F})}$. Since the diagram in (21) commutes, we have $\omega \circ e = \widehat{c} \circ \omega$. Consequently, $c := \omega \circ e \circ \ell = \widehat{c} \circ \omega \circ \ell = \widehat{c} \circ \mathrm{id}_{\mathsf{Att}(\mathcal{F})} = \widehat{c}$, which proves that $\mathsf{A}_\ell = c(\mathsf{Att}(\mathcal{F})) = \widehat{c}(\mathsf{Att}(\mathcal{F})) = \mathsf{A}$. ■

5.2 Remark As in Remark 2.6, if an order retraction to $\mathsf{RC}(\mathcal{F})$ does not exist, then one can try to construct order retraction to an intermediate poset between $\mathsf{RC}(\mathcal{F})$ and $\mathsf{SC}(\mathcal{F})$, which would give a tesselated Morse decomposition $\mathsf{M}(\mathsf{A}_\ell) \hookrightarrow \mathsf{T}(\mathsf{N}_\ell)$ where $\mathsf{T}(\mathsf{N}_\ell)$ which is finer than $\mathsf{RC}(\mathcal{F})$ but coarser than $\mathsf{T}(\mathsf{N})$. In this case combinatorial model would need to be modified by enlarging the attractor lattice $\mathsf{Att}(\mathcal{F})$. ▪

5.2. **Binary relations on tilings of X.** In Section 4 we discussed an extension of the Birkhoff representation theorem which states that the combinatorial model $\mathsf{K} \twoheadrightarrow \mathsf{L}$ is equivalent to choosing a finite binary relation which serves as a model for the dynamical system $f\colon X \to X$. Given a commutative combinatorial model Theorem 1.5 implies that there exists a binary relation $(\mathcal{X}, \mathcal{F})$ such that

$$
\begin{array}{ccc}
\mathsf{K} & \xleftrightarrow{\;e\;} & \mathsf{N} \\
{\scriptstyle h}\big\downarrow\!\!\downarrow & & \big\downarrow\!\!\downarrow{\scriptstyle \omega} \\
\mathsf{L} & \xrightarrow{\;c\;}\!\!\!\!\rightarrow & \mathsf{A}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathsf{SC}(\mathcal{F}) & \longleftrightarrow & \mathsf{T}(\mathsf{N}) \\
{\scriptstyle i}\big\uparrow{\scriptstyle \subset} & & \big\uparrow{\scriptstyle \pi} \\
\mathsf{RC}(\mathcal{F}) & \longleftarrow\!\!\!\supset & \mathsf{M}(\mathsf{A})
\end{array}
\tag{24}
$$

where we choose $\mathsf{N} \subset \mathsf{ABlock}_{\mathscr{R}}(f)$. Computationally $\mathsf{M}(\mathsf{A})$ cannot be determined in principle. However, we can identify $\mathcal{X}$ with the set of tiles $\mathsf{T}(\mathsf{N})$, and without loss of generality consider $\mathcal{X}$ as a a grid on X, ie. a finite sublattice of $\mathscr{R}(\mathcal{X})$.

In [20], a *weak outer approximation* of $f\colon X \to X$ is defined to be a binary relation $(\mathcal{X}, \mathcal{F})$ on a finite lattice of regular closed subsets of X such that

$$
f(|\xi|) \subset \mathrm{int}\, \left|\Gamma^+(\xi)\right| = \mathrm{int}\, \left|\bigcup_{n \geqslant 0} \mathcal{F}^n(\xi)\right|.
$$

5.3 Theorem Let $(\mathcal{X},\mathcal{F})$ be a binary relation on a finite lattice of regular closed subsets of X. The maps $\omega\colon \mathsf{Invset}^+(\mathcal{F}) \to \mathsf{Att}(\mathcal{F})$ and $|\cdot|\colon \mathsf{Invset}^+(\mathcal{F}) \leftrightarrow \mathsf{N} \subset \mathsf{ABlock}_{\mathscr{R}}(f)$ form a combinatorial model if and only if $\mathcal{F}$ is a weak outer approximation for f. ▪

Proof. For every $\mathcal{U} \in \mathsf{Invset}^+(\mathcal{F})$ we have that $|\mathcal{U}|$ is an attracting block, and therefore $f(x) \in \operatorname{int}|\mathcal{U}|$ for all $x \in |\mathcal{U}|$. By compactness $f(|\xi|) \subset \operatorname{int}|\mathcal{U}|$ for all $\xi \in \mathcal{U}$. Every set of the form $\Gamma^+(\xi)$ is forward invariant, and $|\xi| \subset |\Gamma^+(\xi)|$ so that $f(|\xi|) \subset \operatorname{int}|\Gamma^+(\xi)|$, which proves that $\mathcal{F}$ is a weak outer approximation.

As above, $\mathsf{J}(\mathsf{Invset}^+(\mathcal{F})) \approx \{\Gamma^+(\xi) \mid \xi \in \mathcal{X}\}$. By definition of a weak outer approximation, the evaluation map takes $\Gamma^+(\xi)$ to an attracting block $|\Gamma^+(\xi)|$. Moreover, every $\mathcal{U} \in \mathsf{Invset}^+(\mathcal{F})$ can be written as $\mathcal{U} = \bigcup_{\xi\in\mathcal{U}} \Gamma^+(\xi) = \bigvee \Gamma^+(\xi)$ and $|\mathcal{U}| = \bigcup |\Gamma^+(\xi)| = \bigvee |\Gamma^+(\xi)|$ in $\mathsf{ABlock}_{\mathscr{R}}(f)$, and hence $|\cdot|\colon \mathsf{Invset}^+(\mathcal{F}) \to \mathsf{ABlock}_{\mathscr{R}}(f)$. Since $\Gamma^+(\xi) = \Gamma^+(\xi')$ iff $\xi = \xi'$ and each $|\xi| \neq \varnothing$, the evaluation map is injective. Since the evaluation map $|\cdot|$ is a lattice homomorphism from $\mathsf{Set}(\mathcal{X})$ to $\mathscr{R}(X)$, and $\mathsf{Invset}^+(\mathcal{F})$ is a sublattice of $\mathsf{Set}(\mathcal{X})$, we have that $|\cdot|$ is a lattice isomorphism from $\mathsf{Invset}^+(\mathcal{F})$ onto its image N in $\mathscr{R}(X)$. Therefore N is a finite sublattice of $\mathsf{ABlock}_{\mathscr{R}}(f)$, and we have a combinatorial approximation. ■

5.4 Remark Note that the proof of Theorem 5.3 did not use the map $\omega\colon \mathsf{Invset}^+(\mathcal{F}) \to \mathsf{Att}(\mathcal{F})$. Indeed, it is possible to have a weak outer approximation for which $\mathsf{Att}(\mathcal{F}) = \varnothing$ so that $\mathsf{RC}(\mathcal{F}) = \varnothing$, and no recurrence in f is detected. This reinforces the need to have a commutative combinatorial model to relate the combinatorial attractors in $\mathcal{F}$ to attractors of f. ▪

5.5 Lemma Let $(\mathcal{X},\mathcal{F})$ be a binary relation on a finite lattice of regular closed subsets. If $\omega\colon \mathsf{Invset}^+(\mathcal{F}) \to \mathsf{Att}(\mathcal{F})$ and $|\cdot|\colon \mathsf{Invset}^+(\mathcal{F}) \leftrightarrow \mathsf{N} \subset \mathsf{ABlock}_{\mathscr{R}}(f)$ form a commutative combinatorial model, then for each $\mathcal{U} \in \mathsf{Invset}^+(\mathcal{F})$ we have $\omega(|\omega(\mathcal{U})|) = \omega(|\mathcal{U}|)$. ▪

Proof. Let c be the connecting homomorphism in Diagram (22). Then we have $\omega(|\mathcal{U}|) = c(\omega(\mathcal{U}))$. Since $\omega(\mathcal{U})$ is forward invariant, $\omega(\omega(\mathcal{U})) = \omega(\mathcal{U})$ implies $\omega(|\omega(\mathcal{U})|) = c(\omega(\mathcal{U})) = \omega(|\mathcal{U}|)$. ■

The evaluation homomomorphism yields the weak outer approximation property. The connecting homomorphism adds the additional property that $\omega(|\omega(\mathcal{U})|) = \omega(|\mathcal{U}|)$. Outer approximations certainly satisfy the latter, but in general the relations coming from commutative combinatorial models lie in between weak outer approximations and (strong) outer approximations.

6. **Computational Examples.**

In this section, we conclude with some computational examples that illustrate the applications of the concepts described in the previous sections. All of the computations were performed using the CDS software [17], which includes an implementation of the Order Retraction Algorithm in Section 2.

6.1. **Outer approximation of a nonlinear Leslie model.**

The first example arises from the map $f:\mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} (\theta_1 x_1 + \theta_2 x_2)e^{-\phi(x_1+x_2)} \\ p x_1 \end{bmatrix}, \tag{25}$$

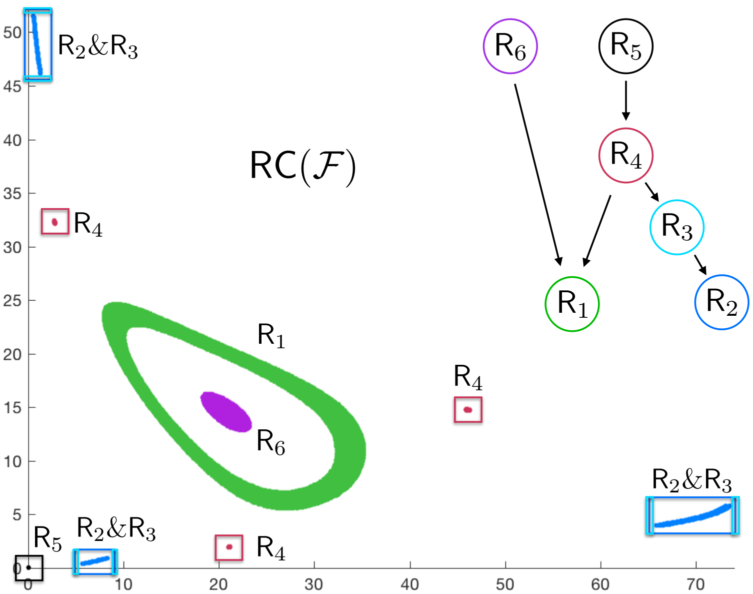
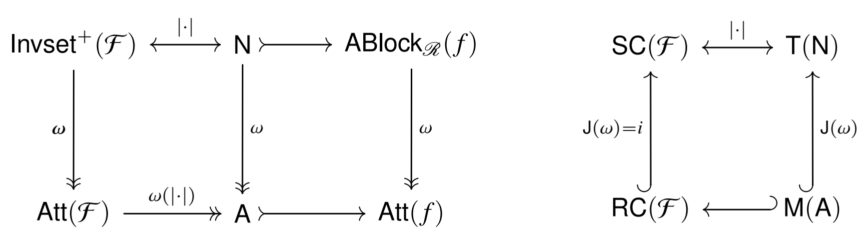


Figure 2. Poset structure of the recurrent components of an outer approximation of the Leslie map (25). Each labeled region is a recurrent component.

where we choose parameters $\theta_1 = 20.0$, $\theta_2 = 20.0$, $\phi = 0.1$, and $p = 0.7$. The phase space is taken to be $X = [0, 74] \times [0, 52]$, which is a forward invariant region. This map is an overcompensatory Leslie population model that has been shown to exhibit a wide variety of dynamical behavior by Ugarcovici and Weiss [29]. For this reason, this multiparameter system is examined in [1] as an in-depth demonstration of computational Conley theory. Using [17], a (rigorous) outer approximation $\mathcal{F}$ is computed for f on X, and the poset structure of the recurrent components $\mathsf{RC}(\mathcal{F})$ is shown in Figure 2. Since we have an outer approximation, a non-cyclic grid element cannot contain any (chain) recurrent dynamics of f. The results of [16] imply that there is a Morse decomposition $\pi\colon \mathsf{M} \hookrightarrow \mathsf{RC}(\mathcal{F})$ where each Morse set in M corresponds to the maximal invariant set of the realization of a recurrent component. Note that it is possible for the realization of a recurrent component to have an empty maximal invariant set.

Since we have an outer approximation, we recall the Diagram (7)



In this example, $\mathsf{SC}(\mathcal{F})$ has 16,343,562 elements; the 6 recurrent components contain 433,654 boxes. We run the algorithm in Theorem 2.4, implemented in [17], and verify that an order retraction $\mathsf{SC}(\mathcal{F}) \twoheadrightarrow \mathsf{RC}(\mathcal{F})$ exists so that $\mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{RC}(\mathcal{F}) \leftrightarrow \mathsf{T}(\mathsf{N}_\ell)$ is a tesselated Morse decomposition by Equation (23). Recall by Lemma 5.1 that $\mathsf{A}_\ell = \mathsf{A}$. The

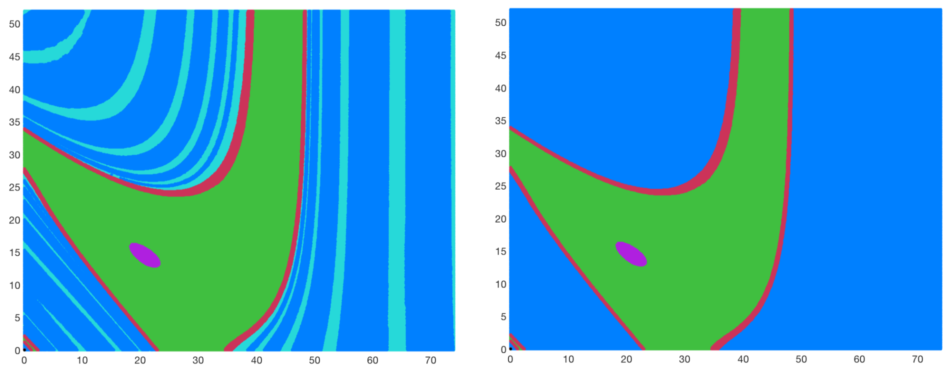


FIGURE 3. Left: Morse tiles of tesselated Morse decompositions $\mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{RC}(\mathcal{F}) \leftrightarrow \mathsf{T}(\mathsf{N}_\ell)$. Right: $\mathsf{M}(\mathsf{A}_{\amalg}) \leftrightarrow \mathsf{RC}_{\amalg}(\mathcal{F}) \leftrightarrow \mathsf{T}((\mathsf{N}_{\amalg})_\ell)$ where $\mathsf{RC}(\mathcal{F})$ is coarsened to $\mathsf{RC}_{\amalg}(\mathcal{F})$ by the retraction of R_3 onto R_2.

tiles in $\mathsf{T}(\mathsf{N}_\ell)$ which correspond to each recurrent component are shown in Figure 3 (left). However, we do not know whether this is an isomorphic decomposition where $\mathsf{M}(\mathsf{A}) \leftrightarrow \mathsf{RC}(\mathcal{F})$.

To gain more insight, we must identify the recurrent dynamics more precisely. We compute the Conley indices of the realizations of each of the recurrent components, which are labeled labeled by R_k for $k \in \{1,2,3,4,5,6\}$. The method used to compute the Conley indices is described in [1]. The components $\mathsf{R}_1, \mathsf{R}_2, \mathsf{R}_4$, and R_6 have nontrivial Conley indices, which implies that inside each of these regions the maximal invariant set is nonempty. The component R_5 contains the origin, which is a fixed point, so that the maximal invariant set inside this region is also nonempty. The component R_3 has a trivial Conley index, which means that we cannot determine topologically whether the maximal invariant set inside this region is empty or not. However, there is a retraction of $\mathsf{RC}(\mathcal{F}) \twoheadrightarrow \mathsf{RC}_{\amalg}(\mathcal{F})$ where $\mathsf{RC}_{\amalg}(\mathcal{F})$ is the subposet of $\mathsf{RC}(\mathcal{F})$ with R_3 removed. Hence there is a retraction $\mathsf{SC}(\mathcal{F}) \twoheadrightarrow \mathsf{RC}_{\amalg}(\mathcal{F})$. The effect of this retraction is to combine the recurrent components R_2 and R_3 as well as each element q of $\mathsf{SC}(\mathcal{F})$ for which $\mathsf{R}_3 \in \mathrm{pred}_{\mathsf{RC}(\mathcal{F})}(q)$ and $\mathsf{R}_2 \in \mathrm{succ}_{\mathsf{RC}(\mathcal{F})}(q)$, that is the connecting orbits between R_3 and R_2. This defines a new relation $\mathcal{F}_{\amalg}$ for which we can consider the corresponding commutative diagrams.

$$\begin{array}{ccccc} \mathsf{Invset}^+(\mathcal{F}_{\amalg}) & \xleftrightarrow{|\cdot|} & \mathsf{N}_{\amalg} & \rightarrowtail & \mathsf{ABlock}_{\mathscr{R}}(f) \\ \omega \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \omega & & \downarrow\!\!\downarrow \omega \\ \mathsf{Att}(\mathcal{F}_{\amalg}) & \xrightarrow{\omega(|\cdot|)} & \mathsf{A}_{\amalg} & \rightarrowtail & \mathsf{Att}(f) \end{array} \qquad \begin{array}{ccc} \mathsf{SC}(\mathcal{F}_{\amalg}) & \xleftrightarrow{|\cdot|} & \mathsf{T}(\mathsf{N}_{\amalg}) \\ \mathsf{J}(\omega)=i \uparrow & & \uparrow \mathsf{J}(\omega) \\ \mathsf{RC}_{\amalg}(\mathcal{F}) = \mathsf{RC}(\mathcal{F}_{\amalg}) & \hookleftarrow & \mathsf{M}(\mathsf{A}_{\amalg}) \end{array} .$$

Since an order retraction $\mathsf{SC}(\mathcal{F}_{\amalg}) \twoheadrightarrow \mathsf{RC}(\mathcal{F}_{\amalg}) = \mathsf{RC}_{\amalg}(\mathcal{F})$ exists, from Equation (23) and Lemma 5.1 we obtain an isomorphic tesselated Morse decomposition $\mathsf{M}(\mathsf{A}_{\amalg}) \leftrightarrow \mathsf{RC}_{\amalg}(\mathcal{F}) \leftrightarrow \mathsf{T}((\mathsf{N}_{\amalg})_\ell)$, and the tiles in $\mathsf{T}((\mathsf{N}_{\amalg})_\ell)$ are shown in Figure 3[right]. The corresponding attractor lattices are shown in Figure 4.

In [13], algorithms are developed to efficiently compute piecewise-constant Lyapunov functions for a system that closely approximate continuous Lyapunov functions for a Morse decomposition. These algorithms also begin with an outer approximation and computation of the recurrent components. The Leslie model with the same parameters as above is used as a benchmark computation on a grid of approximately 19 miilion rectangular tiles. To compute a piecewise-constant approximate Lyapunov function for the Morse decomposition $\mathsf{M}(\mathsf{A}_{\amalg})$ efficiently, Theorem 5.10 in [13] basically requires that $\mathsf{M}(\mathsf{A}) \leftrightarrow \mathsf{T}(\mathsf{N})$ be an isomorphic tesselated Morse decomposition. The algorithm presented in Section 2 along with the Conley index provides a tool to construct such an isomorphic tesselated Morse decomposition as shown above.

This example demonstrates the strong results that can be obtained from an outer approximation. It also illustrates how the overdetermination of the recurrent dynamics that can result from an outer approximation can often be remedied by considering order retractions onto recurrent components which are known to contain nontrivial maximal invariant sets via topological methods.

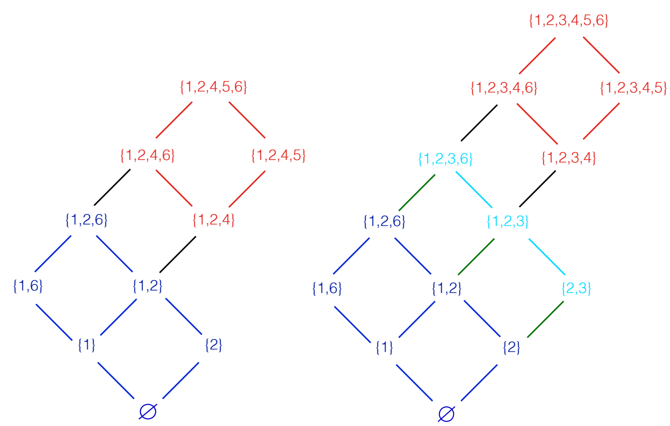


FIGURE 4. Left: Lattice $\mathsf{O}(\mathsf{RC}(\mathcal{F}_{\amalg}))$. Right: Lattice $\mathsf{O}(\mathsf{RC}(\mathcal{F}))$.

6.2. **Combinatorial models for flows.** Consider the flow generated by the solution of an ODE $\dot{x} = f(x)$ in a polygonal region X of $\mathbb{R}^n$. Suppose X is either invariant or an attracting block for the flow, and $\mathcal{X}$ indexes a grid on X composed of convex polygonal tiles, such as a triangulation or a cubical grid, so that the intersection of any pair of tiles is a boundary face of each. In [3], a computational method is described that builds a combinatorial model. First define an equivalence relation on $\mathcal{X}$ as follows. Suppose $\xi, \eta \in \mathcal{X}$ for which the polygons $|\xi|, |\eta|$ satisfy $B = |\xi| \cap |\eta| \neq \emptyset$, and this intersection is an $(n-1)$-dimensional facet. Define $\xi \sim \eta$ if the vector field $f(x)$ is not transverse to B at some point $x \in B$. Also define $\xi \sim \xi$ for all $\xi \in \mathcal{X}$. The realizations of the equivalence classes form a new polygonal grid that tiles X and is indexed by $\mathcal{X}/_{\sim}$. The new polygonal tiles have the property that the internal boundary facets of dimension $n-1$ are transverse to the vector field at every point. Define the relation $\mathcal{F}$ on $\mathcal{X}/_{\sim}$ by $(\xi, \eta) \in \mathcal{F}$ if $|\xi|, |\eta|$ are polygons whose intersection is an $(n-1)$-dimensional facet on which the vector field points out of $|\xi|$ and into $|\eta|$, see Figure 5. As described in [3], the polygonal tiles defined in this way are each isolating blocks for the flow, and for each tile the Conley index can be computed for the maximal invariant sets inside using the direction of the vector field on the $(n-1)$-dimensional faces.

The binary relation $(\mathcal{X}/_{\sim}, \mathcal{F})$ defined above is generally not an outer approximation of the time-T map ϕ_T of the flow for *any* $T > 0$. However, due to the nature of the

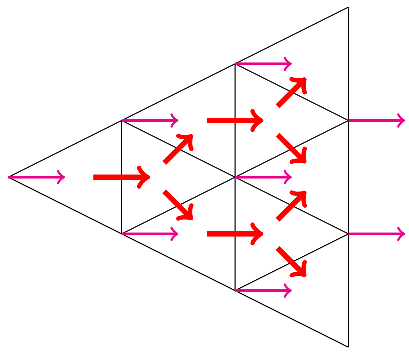


FIGURE 5. Example of a combinatorial model on triangular tiles generated from the flow of a vector field. The vector field is horizontal and plotted at vertices of the triangulation in magenta. The red arrows indicate the resulting binary relation $\mathcal{F}$.

construction, the evaluation map takes $\mathsf{Invset}^+(\mathcal{F})$ into $\mathsf{ABlock}_{\mathscr{R}}(\phi_T)$ for *every* $T > 0$ so that $\mathcal{F}$ is a weak outer approximation [3]. Hence for each $T > 0$ we have

$$\begin{array}{ccccc} \mathsf{Invset}^+(\mathcal{F}) & \xleftrightarrow{|\cdot|} & \mathsf{N} & \rightarrowtail & \mathsf{ABlock}_{\mathscr{R}}(\phi_T) \\ \Big\downarrow \omega & & \Big\downarrow \omega & & \Big\downarrow \omega \\ \mathsf{Att}(\mathcal{F}) & & \mathsf{A} & \rightarrowtail & \mathsf{Att}(\phi_T) \end{array} \tag{26}$$

so that $\mathcal{F}$ induces a combinatorial model for every time-T map of the flow by Theorem 5.3.

As in Remark 5.4, since $\mathcal{F}$ need not be an outer approximation, some recurrent behavior can be missed so that the attractor lattice $\mathsf{Att}(\mathcal{F})$ may not reflect true underlying dynamics. Indeed under this scheme a chain recurrent component of the system that lies entirely within a single polygon, such as an equilibrium point, will often not be represented in the recurrent components of $\mathcal{F}$.

Proceeding as in Section 5.1, one can build a commutative combinatorial model which does represent true underlying dynamics, but the information obtained from it may be coarser than that obtained from an outer approximation of a time-T map. However, the latter is difficult to obtain and is computationally expensive due to the rigorous integration required [22, 18, 14, 28, 27, 25]. The relation $\mathcal{F}$ is computed directly from the vector field without integration, but there can be some computational geometry required to obtain a good representation of the dynamics, see [3].

One can often modify $\mathcal{F}$ to obtain a finer represeantation; for example, one could rigorously locate isolated zeros of the vector field and add a self-loop in $\mathcal{F}$ to each polygon containing an equilibrium point identified this way. Another consideration is that minimal elements of $\mathsf{SC}(\mathcal{F})$ must have realizations that are attracting blocks, which necessarily contain nontrivial recurrent sets in the underlying system, and hence each minimal strong component can be characterized as a recurrent component.

In the next section we explore an example for which this type of combinatorial model arises naturally for a specific class of vector fields for which the grid elements in the phase space can be chosen as rectangular boxes. First we consider a very simple example to illustrate some of the ideas.

6.1 Example Consider the gradient flow of the differential equations $\dot{x} = x(1-x)$ and $\dot{y} = y(y-1)$ on the unit square. There are four equilibria: $e_1 = (1,0), e_2 = (0,0), e_3 = (1,1)$, and $e_4 = (0,1)$, and the finest Morse decomposition is $\{e_1, e_2, e_3, e_4\} = \mathsf{M} \leftrightarrow \{1,2,3,4\}$ with order $1 < 2, 1 < 3, 2 < 4, 3 < 4$, but $2\|3$.

Now consider the tiling $\mathcal{X}$ of the unit square by nine square tiles as shown in Figure 6[left]. The flow is transverse to the interior edges of this tiling, and hence we have a weak outer approximation $\mathcal{F}$ given by the red arrows in the figure. Finally, suppose that we know partial information about the recurrent dynamics so that tiles labeled p_1, p_2, p_4 are given self-loops in $\mathcal{F}$. Then $\mathsf{RC}(\mathcal{F}) \approx \mathsf{P} = \{p_1, p_2, p_4\}$ and $\mathsf{SC}(\mathcal{F}) \approx \mathsf{Q}$ where Q is the poset of all nine tiles. Every element in $\mathsf{Q}\backslash i(\mathsf{P})$ has a unique P-predecessor and a unique P-successor. By Corollary 2.5, we have at least two different order retractions in which each element of $\mathsf{Q}\backslash i(\mathsf{P})$ is mapped to its unique P-successor (Figure 6[middle]) or to its unique P-predecessor (Figure 6[right]) respectively. Taking the maximal invariant sets in each of the resulting regions yields two different Morse decompositions $\mathsf{M}_{13} = \{E_{31}, e_2, e_4\} \leftrightarrow \{1,2,4\}$ with order $1 < 2 < 4$ and $\mathsf{M}_{43} = \{e_1, e_2, E_{43}\} \leftrightarrow \{1,2,4\}$ with order $1 < 2 < 4$ where E_{31} denotes the union of the equilibria e_3 and e_1 along with the connecting orbit between them, and E_{43} denotes the union of the equilibria e_4 and e_3 along with the connecting orbit between them. So the different order retractions result in different Morse decompositions, neither or which is the finest Morse decomposition because the tile q_4 is not a recurrent component of $\mathcal{F}$. ▪

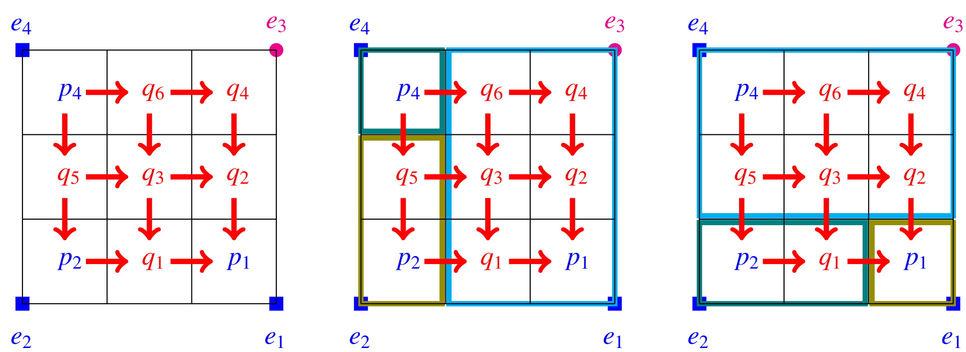
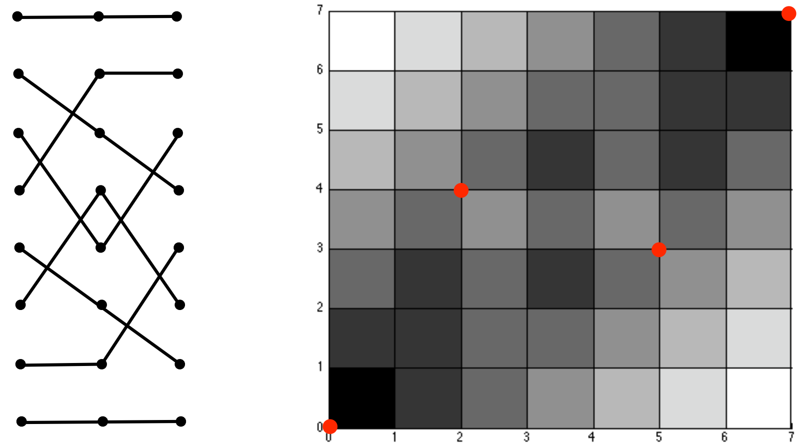


FIGURE 6. Left: Weak outer approximation $(\mathcal{X}, \mathcal{F})$ of the flow in Example 6.1. The tiles labeled in blue as p_1, p_2, p_3 are recurrent, each mapping to itself under $\mathcal{F}$. Middle: order retraction onto the recurrent components by mapping to unique successor. Right: order retraction onto the recurrent components by mapping to unique predecessor.

6.3. **Morse decompositions for parabolic recurrence vector fields.** Consider the flow defined by the system

$$\dot{x}_i = R_i(x_{i-1}, x_i, x_{i+1}) \quad \text{for } i \in \mathbb{Z},$$

where R_i are smooth, bounded functions with $R_{i+d} = R_i$ for some $d > 0$. Moreover, assume each R_i is *parabolic*, i.e. $\partial_1 R_i > 0$ and $\partial_3 R_i > 0$. Finally assume that a set of equilibrium points for (R_i) are known and given as sequences of the form $y_{i+kd} = y_i$. These equiilibria



(a) (b)

FIGURE 7. Left: known equilibrium solutions. Right: the associated tiling of the phase space where sets of boxes of the same color connected by edges are the tiles. The red dots are equilibria of the flow, and the boxes surrounding an equilibrium are also combined into one tile. Solutions flow across boundary edges from lighter colored tiles to darker, as the vector field (R_i) is transverse to these edges, cf. [30].

add additional structure to the problem. In Figure 7 below we show a relation that applies to any system with $d = 2$ and the set of equilibria given by the picture on the left. Due to the parabolic nature of the system, the known equilibrium solutions yield a tiling $\mathcal{X}$ of the phase space by boxes where the vector field is often transverse on edges of the tiles. These systems are studied in general in [12], and the specific example in Figure 7 is described in [30]. We refer the reader to these references for more details; see also [15]. Combining neighboring boxes along edges for which the vector field is nontransverse as in Section 6.2, we obtain a tiling $\mathcal{X}/_\sim$ as shown in Figure 7[right]. The tiles are determined by sets of boxes of the same color connected by edges. The red dots are equilibria of the flow, and the boxes surrounding an equilibrium are also combined into one tile. The flow is from lighter colored tiles to darker, as the vector field is transverse to such edges. Figure 8 shows the directed graph that defines $\mathcal{F}$; the square, blue nodes should have a self-loop since these are known to be recurrent since they contain an equilibrium point. Recall that the tiles defined in this way are each isolating blocks for the flow, and for each tile the Conley index can be computed for the maximal invariant sets inside using the direction of the vector field on the edges.

As in Section 6.2, $\mathcal{F}$ is a weak outer approximation of the time-T map of the flow for every $T > 0$. When the algorithm in Section 2 is applied to this example, an order retraction $\mathsf{Q} = \mathsf{SC}(\mathcal{F}) \twoheadrightarrow \mathsf{RC}(\mathcal{F}) = \mathsf{P}$ does not exist because there are several elements of Q that have multiple P-predecessors and multiple P-successors. The sets P and Q can be viewed in Figure 8 where Q is the set of all nodes, and P consists of the square, blue nodes labeled $\{p_1, p_2, p_3, p_4\}$. The arrows indicate the partial order on the nodes in Q.

Since there is no order retraction, we proceed as in Remark 2.6 and add elements to P until an order retraction exists. This can be done in many ways, but we claim that adding

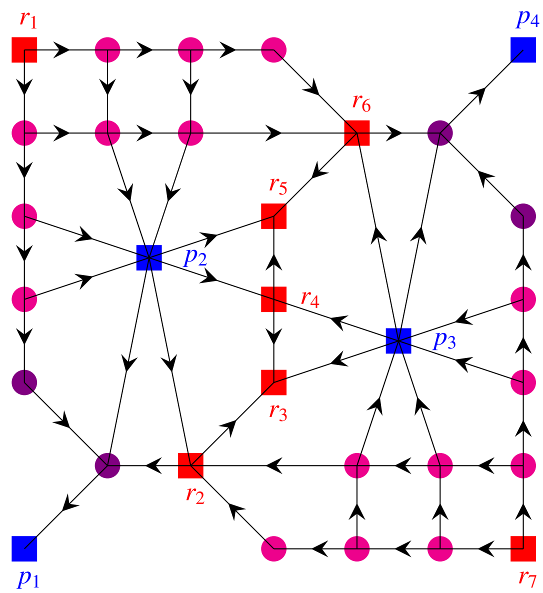


FIGURE 8. The binary relation $\mathcal{F}$ on the tiles $\mathcal{X}/_\sim$. The square, blue nodes should have a self-loop since these are known to be recurrentsss since they contain an equilibrium point.

the square, red nodes labeled $\{r_1,\dots,r_7\}$ to P results in a smallest extension of P for which an order retraction exists. Since r_3, r_5 are minimal nodes with multiple P-predecessors, they both must be added to P to obtain an order retraction. Notice that topologically, these tiles are minimal attracting blocks and therefore must contain nontrivial attractors. Now r_4 has multiple P-successors r_3, r_5 and multiple P-predecessors p_2, p_3 with no elements of Q lying between, so r_4 must be added to P to obtain an order retraction. Note that the Conley index of the corresponding tile in the phase space is that of a saddle point, and hence this tile must contain a nontrivial invariant set. The node r_2 has multiple P-predecessors p_1, r_3 and one P-predecessor p_2. However, r_2 cannot be retracted to p_2 because no retraction of the magenta nodes in the lower, right corner would be order-preserving in that case. Hence r_2, and by symmetry r_6, must both be added to P. Note that the tiles corresponding to r_2 and r_6 both have the Conley index type of a saddle point and must both contain a nontrivial invariant set. Finally, the magenta nodes each have multiple P-successors but no P-predecessor. The simplest way to obtain an order retraction is then to add nodes r_1, r_7 to P. Then Corollary 2.5 applies so that a retraction is given as follows. The magenta nodes in the upper left corner are mapped to r_1, the magenta nodes in the lower right corner are mapped to r_7, the violet nodes in the lower left corner map to p_1, and the violet nodes in the upper right corner map to p_4. The final poset RC for which we have a tesselated Morse decomposition $\mathsf{M}(\mathsf{A}_\ell) \hookrightarrow \mathsf{RC} \leftrightarrow \mathsf{T}(\mathsf{N}_\ell)$ is shown in Figure 9. Note that the tiles r_1 and r_7 may not contain a nontrivial maximal invariant set, since it is not forced topologically. In that case, the addition of these recurrent elements, while necessary to obtain a commutative combinatorial model, would not produce new attractors in the underlying dynamics.

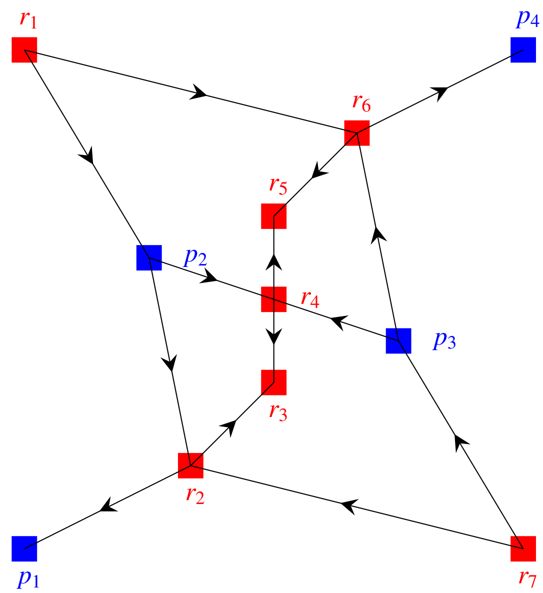


FIGURE 9. The poset RC for which there exists a tesselated Morse decomposition $\mathsf{M}(\mathsf{A}_\ell) \hookrightarrow \mathsf{RC} \leftrightarrow \mathsf{T}(\mathsf{N}_\ell)$.

6.4. **Final remarks.** As stated in the introduction, there has been much recent work on developing methods using set-based computations to analyze global dynamics. The basic tool is a finite binary relation $\mathcal{F}$ that represents the dynamics combinatorially. Reasonable success has been attained in the context of nonlinear systems generated by maps, and some progress has been made on the more technically challenging problem of systems generated by differential equations. The focus has been on using topological methods based on the Conley index and the study of changes with respect to parameter in mutliparameter systems.

In this paper, we have developed an algorithm that takes as input a lattice monomorphism from the forward invariant sets of a relation $\mathcal{F}$ to attracting blocks of the underlying dynamics and produces a tesselated Morse decomposition for the underlying dynamics. These methods use only lattice and order theory, independently from topological methods, and may be applied in a variety of contexts. For example, similar ideas involving the lattice structure of $\mathsf{Att}(\mathcal{F})$ and a mapping into attracting blocks have been used in [11] to analyze the dynamics of switching systems for regulatory networks. The methods developed here could also be used in non-rigorous settings, such as to build and analyze combinatorial models directly from data. The interplay between topological index methods and order theory methods, as illustrated in the example of Section 6.3, also presents an interesting avenue of future research.

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Appendix A. **The category of finite binary relations.** The relationship between the lattice $\mathsf{Att}(\mathcal{F})$ and the relation $\mathcal{F}$ can be formulated in categorical language. The binary relations on a finite set $\mathcal{X}$ form a category which is denoted by **FBRel**. A morphism between finite binary relations $\mathcal{F} \subset \mathcal{X} \times \mathcal{X}$ and $\mathcal{F}' \subset \mathcal{X}' \times \mathcal{X}'$ is a *relation-preserving* mapping $\phi\colon (\mathcal{X},\mathcal{F}) \to (\mathcal{X}',\mathcal{F}')$ that satisfies the property

$$(\xi,\eta) \in \mathcal{F} \quad \text{implies that} \quad \big(\phi(\xi),\phi(\eta)\big) \in \mathcal{F}'. \tag{27}$$

A.1. **The attractor functor.** By **FDLat** we denote the category of finite distributive lattices. For every object $\mathcal{F}$ in **FBRel** define the object $\mathsf{Att}(\mathcal{F})$ in **FDLat** and for every relation-preserving mapping $\phi\colon (\mathcal{X},\mathcal{F}) \to (\mathcal{X}',\mathcal{F}')$ define the mapping $\mathsf{Att}(\phi)\colon \mathsf{Att}(\mathcal{F}') \to \mathsf{Att}(\mathcal{F})$ given by

$$\mathsf{Att}(\phi)(\mathcal{A}') := \omega\big(\phi^{-1}(\mathcal{A}'),\mathcal{F}\big) \in \mathsf{Att}(\mathcal{F}). \tag{28}$$

A.1 Lemma Let $\mathcal{U}' \in \mathsf{Invset}^+(\mathcal{F}')$, then $\omega\big(\phi^{-1}(\mathcal{U}'),\mathcal{F}\big) = \omega\big(\phi^{-1}(\omega(\mathcal{U}',\mathcal{F}')),\mathcal{F}\big)$. ▪

Proof. By construction $\phi^{-1}(\omega(\mathcal{U}')) \subset \phi^{-1}(\mathcal{U}')$. Suppose $\xi \in \phi^{-1}(\mathcal{U}')\backslash\phi^{-1}(\omega(\mathcal{U}'))$ is a cyclic vertex. Then, $\phi(\xi) \in \mathcal{U}'\backslash\omega(\mathcal{U}')$. Since ϕ is relation-preserving also $\phi(\xi)$ is cyclic, which is a contradiction. Therefore, $\phi^{-1}(\mathcal{U}')\backslash\phi^{-1}(\omega(\mathcal{U}'))$ has no cyclic vertices and consequently $\omega\big(\phi^{-1}(\mathcal{U}')\big) = \omega\big(\phi^{-1}(\omega(\mathcal{U}'))\big)$. ∎

A.2 Proposition $\mathsf{Att}\colon \textbf{FBRel} \Longrightarrow \textbf{FDLat}$ defines a contravariant functor and will be referred to as the attractor functor. ▪

Proof. By construction $\mathsf{Att}(\mathcal{F})$ is a finite distributive lattice for every binary relation $\mathcal{F}$. It remains to show that Att acts contravariantly with respect to morphisms. For the identity morphism we have that $\omega(\mathrm{id}^{-1}(\mathcal{A})) = \mathcal{A}$.

Consider a morphism $\phi\colon (\mathcal{X},\mathcal{F}) \to (\mathcal{X}',\mathcal{F}')$. The next step is to show that $\mathsf{Att}(\phi)\colon \mathsf{Att}(\mathcal{F}') \to \mathsf{Att}(\mathcal{F})$ defines a lattice homomorphism. Let $\mathcal{U}'$ be a forward invariant set for $\mathcal{F}'$ so that $\mathcal{F}'(\mathcal{U}') \subset \mathcal{U}'$. Define $\mathcal{U} = \phi^{-1}(\mathcal{U}')$, then $\mathcal{U}$ is forward invariant for $\mathcal{F}$. Indeed, let $\xi \in \mathcal{U}$, $\xi' = \phi(\xi)$ and let $\eta \leftarrow \xi$. Then there exists $\eta = \xi_k \in \mathcal{F}(\xi_{k-1}) \in \cdots \in \mathcal{F}(\xi)$. Under ϕ this implies that $\eta' = \phi(\eta) = \phi(\xi_k) \in \mathcal{F}'(\phi(\xi_{k-1})) \in \cdots \in \mathcal{F}'(\phi(\xi)) = \mathcal{F}'(\xi')$, which implies that $\eta' \leftarrow \xi'$, s and thus $\eta' \in \mathcal{U}'$. The latter implies that $\eta \in \phi^{-1}(\mathcal{U}') = \mathcal{U}$, showing that $\mathcal{U}$ is forward invariant. The application $\mathcal{U}' \mapsto \phi^{-1}(\mathcal{U}')$ preserves the lattice operations and defines a lattice homomorphism from $\mathsf{Invset}^+(\mathcal{F}')$ to $\mathsf{Invset}^+(\mathcal{F})$.

To show that $\mathcal{A}' \mapsto \omega(\phi^{-1}(\mathcal{A}'))$ is a lattice homomorphism from $\mathsf{Att}(\mathcal{F}')$ to $\mathsf{Att}(\mathcal{F})$ we argue as follows. For $\mathcal{A}',\tilde{\mathcal{A}}' \in \mathsf{Att}(\mathcal{F}')$ we have:

$$\begin{aligned}\mathsf{Att}(\phi)(\mathcal{A}' \cup \tilde{\mathcal{A}}') &= \omega\big(\phi^{-1}(\mathcal{A}' \cup \tilde{\mathcal{A}}'),\mathcal{F}\big) = \omega\big(\phi^{-1}(\mathcal{A}') \cup \phi^{-1}(\tilde{\mathcal{A}}'),\mathcal{F}\big)\\ &= \omega\big(\phi^{-1}(\mathcal{A}'),\mathcal{F}\big) \cup \omega\big(\phi^{-1}(\tilde{\mathcal{A}}'),\mathcal{F}\big) = \mathsf{Att}(\phi)(\mathcal{A}') \cup \mathsf{Att}(\phi)(\tilde{\mathcal{A}}')\end{aligned}$$

and

$$\begin{aligned}\mathsf{Att}(\phi)(\mathcal{A}' \wedge \tilde{\mathcal{A}}') &= \omega\big(\phi^{-1}(\omega(\mathcal{A}' \cap \tilde{\mathcal{A}}',\mathcal{F}')),\mathcal{F}\big) \text{ by Lemma A.1}\\ &= \omega\big(\phi^{-1}(\mathcal{A}' \cap \tilde{\mathcal{A}}'),\mathcal{F}\big) = \omega\big(\phi^{-1}(\mathcal{A}') \cap \phi^{-1}(\tilde{\mathcal{A}}'),\mathcal{F}\big)\\ &= \omega\big(\omega(\phi^{-1}(\mathcal{A}'),\mathcal{F}) \cap \omega(\phi^{-1}(\tilde{\mathcal{A}}'),\mathcal{F}),\mathcal{F}\big) \text{ by [20, Prop. 2.8]}\\ &= \mathsf{Att}(\phi)(\mathcal{A}') \wedge \mathsf{Att}(\phi)(\tilde{\mathcal{A}}'),\end{aligned}$$

which proves that $\mathsf{Att}(\phi)$ is a lattice homomorphism.

Consider the composition $\psi \circ \phi$, with $\phi\colon (\mathcal{X},\mathcal{F}) \to (\mathcal{X}',\mathcal{F}')$ and $\psi\colon (\mathcal{X}',\mathcal{F}') \to (\mathcal{X}'',\mathcal{F}")$ relation preserving. By definition $\mathsf{Att}(\psi \circ \phi) = \omega \circ \phi^{-1} \circ \psi^{-1}$. By Lemma A.1 we have

$$\begin{aligned}\mathsf{Att}(\psi\circ\phi)(\mathcal{A}'') = (\omega\circ\phi^{-1}\circ\psi^{-1})(\mathcal{A}'') &= \omega\big(\phi^{-1}(\psi^{-1}(\mathcal{A}''))\big) = \omega\Big(\phi^{-1}\big(\omega(\psi^{-1}(\mathcal{A}''))\big)\Big)\\ &= \mathsf{Att}(\phi)\big(\mathsf{Att}(\psi)(\mathcal{A}'')\big) = \big(\mathsf{Att}(\phi)\circ\mathsf{Att}(\psi)\big)(\mathcal{A}''),\end{aligned}$$

which proves that Att is a contravariant functor. ■

Denote the category of finite posets by **FPoset**. For every object $\mathcal{F}$ in **FBRel** define the object $\mathsf{RC}(\mathcal{F})$ in **FPoset** and for every relation-preserving mapping $\phi\colon (\mathcal{X},\mathcal{F}) \to (\mathcal{X}',\mathcal{F}')$ define the mapping $\mathsf{RC}(\phi)\colon \mathsf{RC}(\mathcal{F}) \to \mathsf{RC}(\mathcal{F}')$ given by $\mathsf{RC}(\phi)([\xi]_{\leftrightarrow}) := [\phi(\xi)_{\leftrightarrow}]$.

A.3 Proposition $\mathsf{RC}\colon$ **FBRel** $\Longrightarrow$ **FPoset** defines a covariant functor and is referred to as the recurrent functor. ▪

Proof. By construction $\mathsf{RC}(\mathcal{F})$ is a finite poset for every $\mathcal{F}$. It remains to show that RC acts as covariantly with respect to morphisms. Let $\phi\colon (\mathcal{X},\mathcal{F}) \to (\mathcal{X}',\mathcal{F}')$ be a morphism. Since ϕ is relation-preserving, $\mathsf{RC}(\phi)$ mapping $([\xi]_{\leftrightarrow})$ to $[\phi(\xi)_{\leftrightarrow}]$ is order-preserving. Given a morphism $\psi\colon (\mathcal{X}',\mathcal{F}') \to (\mathcal{X}'',\mathcal{F}")$, the composition law follows from the definition. ■

A.2. **Reflexive closure.** Let $\mathcal{F}^=$ be the reflexive closure of $\mathcal{F}$ and consider the identity mapping $\phi = \mathrm{id}$ as relation-preserving mapping $\phi\colon (\mathcal{X},\mathcal{F}) \to (\mathcal{X},\mathcal{F}^=)$.

A.4 Lemma The induced homomorphism $\mathsf{Att}(\mathrm{id})\colon \mathsf{Att}(\mathcal{F}^=) \twoheadrightarrow \mathsf{Att}(\mathcal{F})$ is a lattice epimorphism and for $\mathcal{U} \in \mathsf{Att}(\mathcal{F}^=)$, $\mathsf{Att}(\mathrm{id})(\mathcal{U}) = \omega(\mathcal{U})$. ▪

Proof. Let $\mathcal{U} \in \mathsf{Att}(\mathcal{F}^=)$, then, since $\phi = \mathrm{id}$, $\omega(\phi^{-1}(\mathcal{U})) = \omega(\mathcal{U})$. Surjectivity follows by taking $\mathcal{U} = \mathcal{A} \in \mathsf{Att}(\mathcal{F})$. ■

A.5 Lemma $\mathsf{Invset}^+(\mathcal{F}) = \mathsf{Att}(\mathcal{F}^=)$ and $\mathsf{SC}(\mathcal{F}) = \mathsf{RC}(\mathcal{F}^=)$. ▪

Proof. By definition $\mathsf{Att}(\mathcal{F}^=) \subset \mathsf{Invset}^+(\mathcal{F}^=)$. Let $\mathcal{U} \in \mathsf{Invset}^+(\mathcal{F})$, then $\mathcal{F}^=(\mathcal{U}) \subset \mathcal{U}$. On the other hand since $\mathcal{F}^=$ is reflexive we have that $\mathcal{U} \subset \mathcal{F}^=(\mathcal{U})$ which proves that $\mathcal{F}^=(\mathcal{U}) = \mathcal{U}$, and thus $\mathsf{Invset}^+(\mathcal{F}^=) \subset \mathsf{Att}(\mathcal{F}^=)$. The proof of the statement for RC and SC is similar. ■

Lemmas A.4 and A.5 recover the lattice epimorphism $\omega\colon \mathsf{Invset}^+(\mathcal{F}) \twoheadrightarrow \mathsf{Att}(\mathcal{F})$ in (15). If we apply the functor RC to $\mathrm{id}\colon (\mathcal{X},\mathcal{F}) \to (\mathcal{X},\mathcal{F}^=)$, then $\mathsf{RC}(\mathrm{id})([\xi]_{\leftrightarrow}) = [\xi]_{\leftrightarrow}$ defines an order-embedding $\mathsf{RC}(\mathcal{F}) \hookrightarrow \mathsf{RC}(\mathcal{F}^=) = \mathsf{SC}(\mathcal{F})$ by Lemma A.5, as in (16).

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