On the Detection of Simple Points in Higher Dimensions Using Cubical Homology

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Abstract

Simple point detection is an important task for several problems in discrete geometry, such as topology preserving thinning in image processing to compute discrete skeletons. In this paper, the approach to simple point detection is based on techniques from cubical homology, a framework ideally suited for problems in image processing. A (d-dimensional) unitary cube (for a d-dimensional digital image) is associated with every discrete picture element, instead of a point in $\mathcal{E}^d$ (the d-dimensional Euclidean space) as has been done previously. A simple point in this setting then refers to the removal of a unitary cube without changing the topology of the cubical complex induced by the digital image. The main result is a characterization of a simple point p (i.e., simple unitary cube) in terms of the homology groups of the $(3^d - 1)$ neighborhood of p for arbitrary, finite dimensions d.

Index Terms

Simple point, cubical homology, skeleton, digital geometry
EDICS: 2-ANAL

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I. Introduction

In the context of digital topology discrete data sets are associated with geometric objects such as two dimensional images, three dimensional volumes, etc. Typically the data set is obtained by assigning to each pixel, voxel, tetrapus, or any analogous higher-dimensional object a point, e.g. the center point. Connectivity, and in general topology, is assigned to this set of points by applying a particular neighborhood structure. The choice of neighborhood structure determines which topological properties carry over from the continuous to discrete settings. For example, when looking at a discrete image composed of pixels, not all neighborhood definitions (for the foreground and the background) will allow for a discrete equivalent to the Jordan curve theorem.

Given a particular set of points $\mathcal{P}$ and fixed neighborhood structure, one can ask what happens to the induced topology when one of the points $p \in \mathcal{P}$ is removed. A point p is *simple* in $\mathcal{P}$, if the topology associated with $\mathcal{P}$ is equivalent to the topology associated with $\mathcal{P} \setminus \{p\}$ (a precise definition is presented below). Detecting simple points is of crucial importance in thinning applications for example, where a discrete representation of an object gets reduced to its topologically equivalent skeleton. In the simplest (continuous) Euclidean case, the skeleton or medial axis (in the sense of Blum [1]) is the set of shock points emanating from an inward moving object boundary traveling with unit speed. If the skeleton is augmented with arrival time information perfect boundary reconstruction can be achieved. Skeletons can for example be used as shape descriptors for object recognition, for object compression, to find centerlines of objects, etc. See [2], [3] for a review of skeletonization algorithms.

Given its importance it is not surprising that simple point detection has been studied in the context of digital topology. Since most applications in image processing deal with two- or three dimensional images, identification of simple points in this setting is well understood. However, three dimensional image sequences (e.g., of a beating heart) demand simple point detection algorithms for four-dimensional spaces (see [4] for a four-dimensional simple point result). Furthermore, problems for higher space dimensions are easily conceivable: the recent work by Han et al. [5] requires the detection of simple points during a level set evolution to preserve the topology of an implicitly evolving surface; and the removal of simple points is essential for the efficient implementation of new dimension independent algorithms for computing the homology of maps [6].

The complexity of simple point detection increases with the dimension of the space. If one applies the neighborhood structure known as $(3^d - 1)$-connectivity, then the number of elements in a neighborhood increases exponentially with dimension d. Furthermore, in the case of thinning, simple point decisions often need to be made multiple times for many elements of the data set (removing one simple point may create a new simple point), resulting in a large number of decisions to be taken. Thus, computationally efficient and dimension-independent algorithms for simple point detection are clearly necessary.

This paper develops a characterization of a simple point p of a discrete d-dimensional, binary dataset in terms of the homology groups of the cubical complex induced by the $(3^d - 1)$-neighborhood of p. In contrast to most previous approaches this characterization is dimension-independent and thus facilitates the design of dimension-independent algorithms (for an alternative approach see Pilarczyk [7]). Furthermore, computationally efficient algorithms to compute the resulting simple point condition exist and are freely available. Our approach is most closely related to that of Tourlakis and Mylopoulos [8], [9], but associates a d-dimensional unitary cube (for a d-dimensional digital image) with every discrete picture element, instead of a point in the d-dimensional Euclidean space $\mathcal{E}^d$. The simple point detection scheme proposed in this paper makes use of cubical homology; for a recent treatment see [10].

We now outline the contents of this note. Section II gives proposed simple point conditions whose proofs are given in the Appendix . Section III relates the approach presented in this paper to previous work and compares it in particular to the approach by Tourlakis and Mylopoulos [8], [9]. Our approach is less general, but better suited for the processing of digital datasets. Section IV summarizes the salient points of our methodology and makes some conclusions.

II. Detecting Simple Points by Cubical Homology

This section presents the main results of this paper. We will also aim at giving some intuition for these simple point conditions. For clarity we assume that a d-dimensional image I_d is composed of *unitary d-dimensional cubes*, i.e. translates of $[0,1]^d \subset \mathbb{R}^d$, whose center $p \in \mathbb{Z}^d$ has integer coordinates. Note that the size of the cubes plays no role in determining the topology of the image.

Define the (3^d-1)-neighborhood and the $(2d)$-neighborhood of p as

$$\begin{aligned} N_{3^d-1}(p) &= \{\overline{p} \in \mathbb{Z}^d : \max_{1\le i\le d,\ i\in\mathbb{N}} |p_i - \overline{p}_i| \le 1\}, \\ N_{2d}(p) &= \{\overline{p} \in \mathbb{Z}^d : \sum_{1\le i\le d,\ i\in\mathbb{N}} |p_i - \overline{p}_i| \le 1\} \end{aligned}$$

respectively, where p_i is the i-th coordinate of p. Two points x and y are *n-connected* if $x \in N_n(y)$.

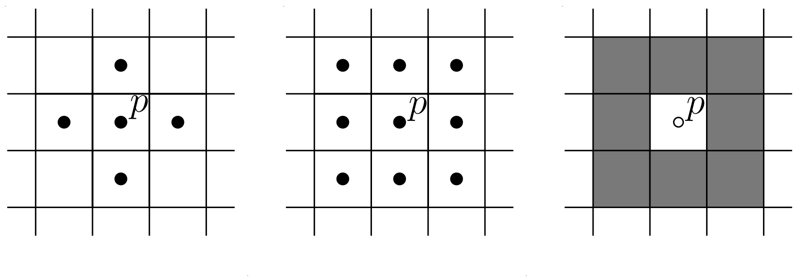
Definition 1 (Cubical Neighborhood)

Let $\mathcal{P} \subset \mathbb{Z}^d$. The cubical neighborhood $CN(p)$ of a point $p \in \mathcal{P}$ is the union of the d-dimensional unitary cubes associated with the points in $N_{3^d-1}(p) \setminus p$, i.e.,

$$CN(p) = \bigcup_{q\in\left(N_{3^d-1}(p)\cap\mathcal{P}\right)\setminus p} C(q).$$

Figure (1) illustrates the different neighborhood concepts in two dimensions

As is indicated in the Introduction, a point is simple if its removal results in an equivalent topological space. Thus the notion of simple is directly tied to the equivalence relation that is being imposed on the



(a) 4-neighborhood of p. (b) 8-neighborhood of p. (c) Cubical neighborhood $CN(p)$ of p

Fig. 1. Two-dimensional neighborhoods.

topological space. At first glance it may appear that being homeomorphic is the most natural equivalence relation. However, this is an extremely restrictive form of equivalence and hence difficult to verify. Furthermore, it is unnecessarily strong; for example, an object is not homeomorphic to its skeleton or medial axis. This suggests the following definition.

Definition 2 (Homotopy-Simple Point)

Let $\mathcal{P} \subset \mathbb{Z}^d$. Let

$$P := \bigcup_{q \in \mathcal{P}} C(q) \quad \text{and} \quad P' := \bigcup_{q \in \mathcal{P} \setminus \{p\}} C(q).$$

A point p is homotopy-simple if P is homotopy equivalent to P'.

Planar examples such as those of Figures (2) and (3) suggest that p is a homotopy-simple point if and only if $CN(p)$ is contractible (i.e. homotopic) to a single point. While in low dimensions this condition is easy to visualize, our goal is to provide an algorithm for the verification of simple points and thus we need a reformulation that is amenable to computation. This can be accomplished using algebraic topology. While homotopy groups provide an algebraic means of determining homotopy type, unfortunately they are notoriously difficult to compute. In contrast, homology groups are effectively computable. Homology assigns to each topological space X a sequence of abelian groups, $H_k(X)$, $k = 0, 1, 2, \ldots$, called *homology groups*.

Homology can be computed by decomposing the space into a finite number of units. In the traditional simplicial homology, these units are simplices, a formalization of the notion of triangulation. In the cubical homology these units are pixels/voxels and their respective vertices, edges and higher-dimensional faces. Cubical homology is ideally suited for digital images, due to its ability to handle d-dimensional cubes directly. Whereas simplicial homology is by now a standard tool of algebraic topology [11], the direct application of cubical homology is much more recent [10], [12].

Definition 3 (Homology-Simple Point)

Let $\mathcal{P} \subset \mathbb{Z}^d$. Let

$$P := \bigcup_{q \in \mathcal{P}} C(q) \quad \text{and} \quad P' := \bigcup_{q \in \mathcal{P} \setminus \{p\}} C(q).$$

A point p is homology-simple if $H_k(P) \cong H_k(P')$ for all $k \geq 0$ where the isomorphisms are induced by inclusion.

It is important to note that if two spaces are homotopy equivalent, then they have the same homology groups; however the converse need not be true. Thus, a homology-simple point is, in general, a weaker concept than a homotopy-simple point. Nevertheless, as we shall demonstrate, for low dimensional settings, i.e. $\mathcal{P} \subset \mathbb{Z}^d$ where $d \leq 3$ the two concepts coincide.

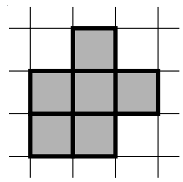
We begin with a dimension independent result.

Theorem 1 (Homology-Simple Point)

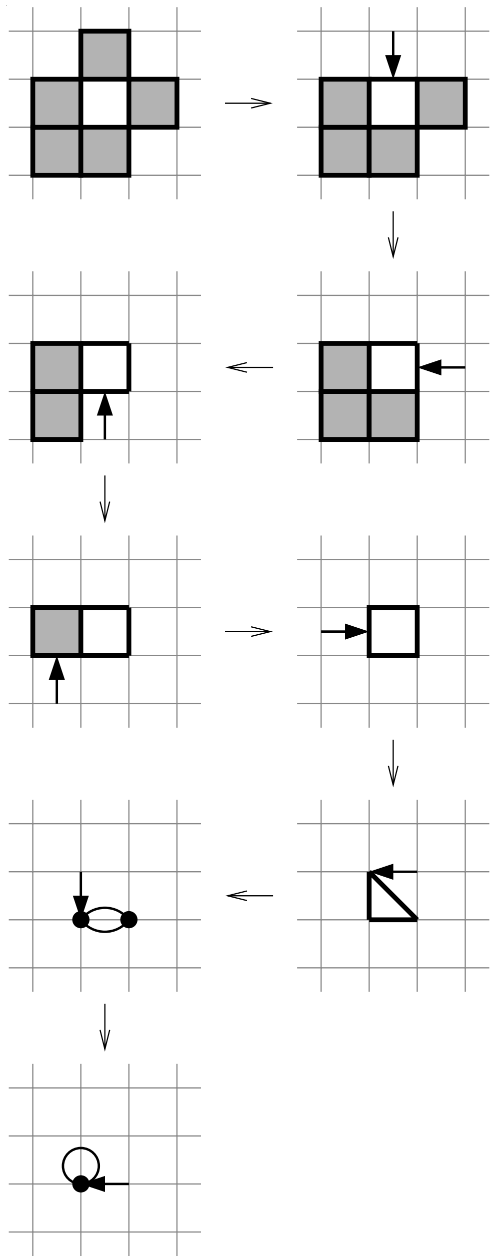
A point p is a homology-simple point in $\mathcal{P}$ if and only if its cubical neighborhood $CN(p)$ is acyclic; that is

$$H_k(CN(p)) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ 0 & \text{if } k \geq 1. \end{cases}$$

The proof is straightforward (it follows from [6, Lemma 7.1] and [7, Lemma 9]), however for the sake

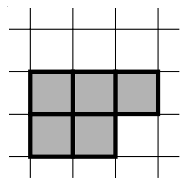


(a) Original cubical complex.

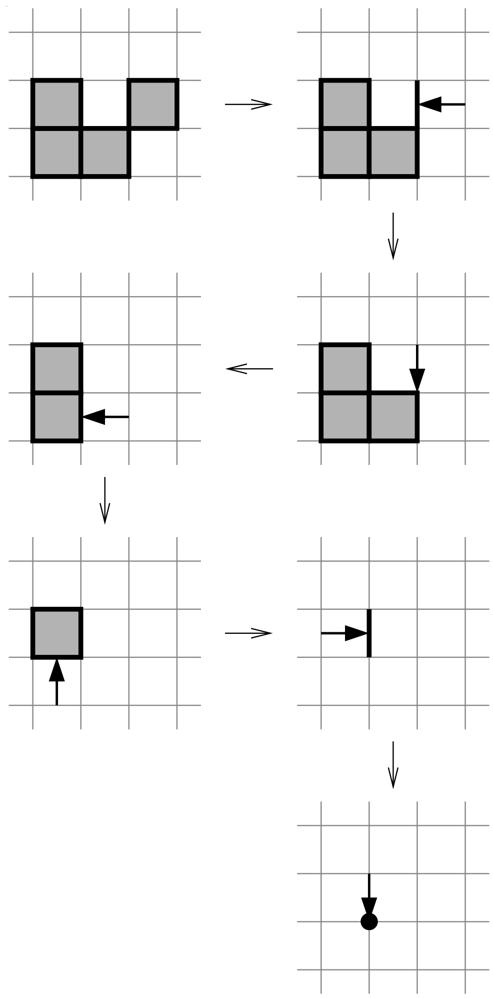


(b) Cubical complex with removed center cube.

Fig. 2. Collapsing a cubical complex. The cyclic case. $\beta_0 = 1$, $\beta_1 = 1$. The point is not simple.



(a) Original cubical complex.



(b) Cubical complex with removed center cube.

Fig. 3. Collapsing a cubical complex. The acyclic case. $\beta_0 = 1$, $\beta_1 = 0$. The point is simple.

of completeness we include it in the Appendix .

A classical result due to H. Poincaré implies that there exist cubical neighborhoods $CN(p)$ which are acyclic, but not contractible. Thus, this result is insufficient to guarantee that a homology-simple point is a homotopy-simple point.

A homology group $H_k(X)$ is *torsion-free* if $H_k(X) \cong \mathbb{Z}^{\beta_k}$. In this case the homology group is completely described by its rank which is referred to as the k-th Betti number and denoted by $\beta_k(X)$. The Betti numbers provide considerable geometric information. For example, given a three-dimensional image data, there are at most three nontrivial homology groups H_0, H_1 and H_2. The number of connected components, tunnels, and voids present in the image are given by the Betti numbers β_0, β_1 and β_2.

Theorem 2 (Betti Number Characterization of Simple Point)

Let $p \in \mathcal{P} \subset \mathbb{R}^d$ and assume $d \leq 4$. A point p is a homology-simple point in $\mathcal{P}$ if and only if

$$\beta_k(CN(p)) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \geq 1. \end{cases}$$

Proof: Let $d = 4$ and assume that $C = [0,1]^4$ is the elementary cube associated with $p \in \mathcal{P} \subset \mathbb{R}^4$. Let $X := CN(p) \cap C$. It is straightforward though technical to check that X is homotopy equivalent to $CN(p)$. Observe that X is a cubical set contained in the boundary of $[0,1]^4$. Since the boundary of $[0,1]^4$ is homeomorphic to S^3, the unit sphere in $\mathbb{R}^4$, $H_*(X)$ is torsion free [13]. Thus, the hypothesis of the theorem is equivalent to the acyclicity of X. Analogous arguments hold for $d < 4$. ■

Real projective space $\mathbb{R}P^2$ provides an example of a topological space whose homology is not torsion free, and hence, not completely described by its Betti numbers. $\mathbb{R}P^2$ can be expressed as a simplicial complex that consists of 10 triangles, 15 edges, and 6 vertices. This implies, by [10, Theorem 11.17] that it can be represented as a cubical complex in the boundary of $[0,1]^5 \subset \mathbb{R}^5$. In particular, homology-simple points for $\mathcal{P} \subset \mathbb{Z}^d$ cannot be characterized by Betti numbers for $d \geq 5$. Thus, the previous theorem is sharp.

Theorem 3 (Homotopy-Simple Point)

Let $p \in \mathcal{P} \subset \mathbb{R}^d$ and assume $d \leq 3$. A point p is a homotopy-simple point in $\mathcal{P}$ if and only if

$$\beta_k(CN(p)) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \geq 1. \end{cases}$$

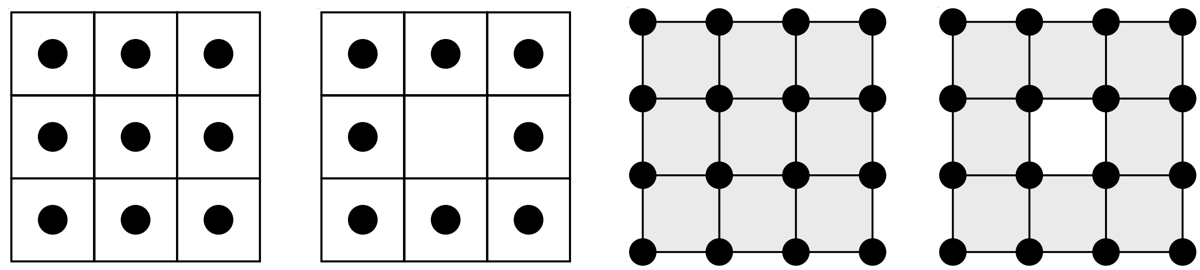
Again, the proof of this result is straightforward but rather technical and appears in the Appendix .

As is suggested above, at least in lower dimensional spaces, Betti numbers and hence homology groups correspond to intuitive geometric concepts like the number of connected components, the number of loops, or the number of enclosed cavities. Thus, having a simple point characterization in terms of homology groups is also useful for applications where topology does not necessarily need to be preserved, i.e. it is relatively straightforward to augment the simple point condition with additional rules based on homology groups to allow for meaningful non-topology-preserving thinning (e.g. to always allow the thinning to a line without the need for dealing with complicated special cases).

III. Relation to Previous Work

Many approaches for simple point detection have been proposed [14]–[16]. While the method in this paper works on the cubical level, many previous approaches work on the point level; see Figure (4) for an illustration.

Simple point algorithms have been based on connected component analysis, computation of Euler characteristics, or template matching, to name a few of the most popular methods. Tabulating simple



(a) One point per pixel before removal of the center pixel.

(b) One point per pixel after removal of the center pixel.

(c) One unitary cube per pixel before removal of the center pixel.

(d) One unitary cube per pixel after removal of the center pixel.

Fig. 4. Removing a pixel in a two-dimensional digital image. Illustration of the point and the unitary cube based approaches.

point configurations is prohibitive in higher dimensions. For a four-dimensional cube there are already 2^{80} possible neighborhood configurations. Many simple point detection approaches are not dimension independent. For example approaches relying on the graph induced by the points describing a discrete dataset and their neighborhood structure. While the dimensional dependency of a simple point condition may be acceptable for computational efficiency (for example, efficient tabulation has been implemented in [17], [18] for three dimensions), it is desirable to have more general, computable conditions for higher dimensions– for example to facilitate easy implementation of dimension independent thinning algorithms. A comprehensive review of related methods is beyond the scope of this paper. The interested reader is referred to the excellent survey articles by Kong and Rosenfeld [19] on digital topology and by McAndrew and Osborne [20] on algebraic methods in digital topology and the references therein. Further, see [3], [21] for the use of algebraic topology in image processing, see [22] for the computation of the Euler number based on cubical homology and [23] for a general overview of computational topology.

The approach for simple point detection proposed in Section II is rooted in algebraic topology and most closely related to the work of Tourlakis and Mylopoulos [9] to which it will be compared in the remainder of this section.

Given a set of points

$$\mathcal{P} = \left\{ \bigcup_i p^i : p^i \in \mathbb{Z}^d \right\},$$

the polyhedron $\Pi(\mathcal{P})$ is made up solely of the points that fulfill

1) $\mathcal{P} \subset \Pi(\mathcal{P})$.
2) If all vertices of the unitary cube u are in $\mathcal{P}$, then $u \subset \Pi(\mathcal{P})$.

Then $\Pi(\mathcal{P})$ is the "smallest" polyhedron covering $\mathcal{P}$. Defining the covering polyhedron this way induces a $(2d)$-connectivity between the points in $\mathcal{P}$. Tourlakis and Myopoulos [9] define the *open star of a point x with respect to the set of points $\mathcal{P}$* as

$$St(x, \mathcal{P}) := \{x\} \cup \bigcup_{\substack{x \text{ vertex of } \overset{\circ}{e} \\ e \subset \Pi(\mathcal{P})}} \overset{\circ}{e} \quad ,$$

where $\overset{\circ}{e}$ is an elementary cell[1]. The *closed star of x with respect to $\mathcal{P}$* is

$$[St(x, \mathcal{P})] := \{x\} \cup \bigcup_{\substack{x \text{ vertex of } e \\ e \subset \Pi(\mathcal{P})}} e \quad ,$$

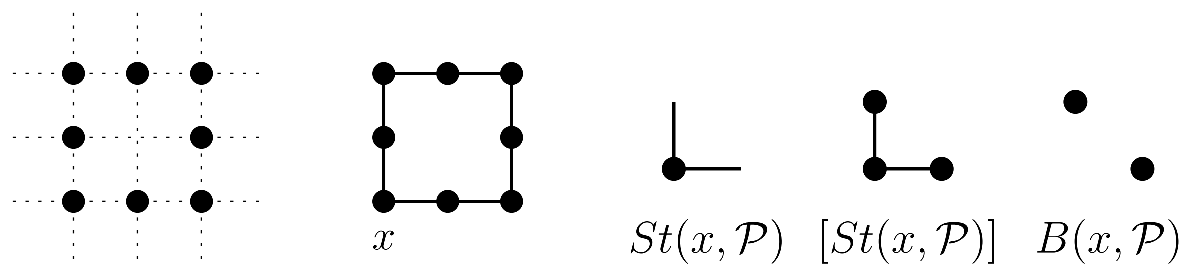
where e is a unitary cube. The *base of x in $\mathcal{P}$* is then defined as

$$B(x, \mathcal{P}) := [St(x, \mathcal{P})] - St(x, \mathcal{P}).$$

Figures (5) and (6) show illustrations of these concepts.

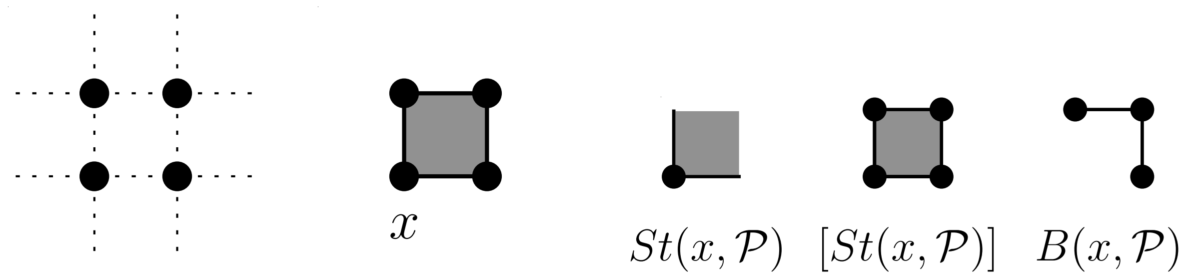
The following proposition [9] is most closely linked to the homology-simple point condition of Theorem 1 of this paper.

[1]The *elementary cell* associated with a general elementary cube $Q = I_1 \times I_2 \times \cdots \times I_d \subset \mathbb{R}^d$ is given by $\overset{\circ}{Q} = \overset{\circ}{I}_1 \times \overset{\circ}{I}_2 \times \cdots \times \overset{\circ}{I}_d$, where $\overset{\circ}{I} := (l, l+1)$ if $I = [l, l+1]$ and $\overset{\circ}{I} := [l]$ if $I = [l, l]$.



(a) Set of points $\mathcal{P}$. (b) Polyhedron $\Pi(\mathcal{P})$. (c) Determining the base $B(x,\mathcal{P})$.

Fig. 5. Using the polyhedron approach to determine if a point x is simple. The loop case. Point x is not simple.



(a) Set of points $\mathcal{P}$. (b) Polyhedron $\Pi(\mathcal{P})$. (c) Determining the base $B(x,\mathcal{P})$.

Fig. 6. Using the polyhedron approach to determine if a point x is simple. The square case. Point x is simple.

Proposition 1

Let $x \in \mathcal{P}$. Assume that the dimension $d \leq 3$. Then the following statements are equivalent:

1) *x is simple,*
2) *$H_k(\Pi(\mathcal{P})) \cong H_k(\Pi(\mathcal{P} \setminus \{x\}))$ for all $k \geq 0$, where the inclusion[2] $\imath : \Pi(\mathcal{P} \setminus \{x\}) \mapsto \Pi(\mathcal{P})$ induces isomorphisms $\imath_k : H_k(\Pi(\mathcal{P} \setminus \{x\})) \mapsto H_k(\Pi(\mathcal{P}))$ for all $k \geq 0$,*
3) *$H_k(B(x,\mathcal{P})) \cong 0$ for all $k \geq 1$, while $H_0(B(x,\mathcal{P})) \cong \mathbb{Z}$.*

Thus, x is simple only if the base $B(x,\mathcal{P})$ of x in $\mathcal{P}$ is simply connected and acyclic.

The homology-simple point condition of Theorem 1 is equivalent to Proposition (1) of [9] in two and three dimensions if we construct the point set $\mathcal{P}$ such that each element of a discrete dataset is represented

[2]The inclusion condition is necessary for $d = 3$ and was proposed by Kong and Rosenfeld [19]. For an illustration of the necessity of this condition see Figure (7) as given in [20].

by all the vertices of its associated unitary cube. This induces a $(3^d - 1)$-connectivity between the unitary cubes associated with the elements of the original discrete dataset. Since we only allow the removal of complete d-dimensional unitary cubes from our cubical complex, the inclusion property of Proposition (1) is automatically satisfied, and the counterexamples to the initial results in [9] as described in [19], [20] no longer hold. Further, Theorem 1 holds for arbitrary finite dimensions, whereas Proposition (1) is only valid in the two-dimensional and the three-dimensional case.

IV. Summary and Conclusions

We introduced two definitions for a simple point: the *homology-simple point* and the *homotopy-simple point*. We showed that in low dimensions ($d \leq 3$) the two concepts coincide. However, the less restrictive concept of a homology-simple point extends to any finite dimension. The dimension-independence of the homology-simple point is in contrast to previous approaches and makes it highly attractive for example for skeletonization or for topology preserving level set methods. The point associated with a unitary cube is a homology-simple point if the cubical complex induced by its $(3^d - 1)$-neighborhood (with the center cube removed) is acyclic. For dimensions $d \leq 4$ this condition may be written in terms of the Betti numbers of the induced cubical complex. The point associated with a unitary cube is a homotopy-simple point if the cubical complex induced by its $(3^d - 1)$-neighborhood (with the center cube removed) is contractible (i.e. homotopic) to a single point.

Dimension-independent algorithms for the computation of Betti numbers in a cubical setting are readily available and facilitate the computation of the simple point condition. These algorithms [17] have been used in dimensions three and higher to implement the simple point detection scheme proposed in this paper.

Appendix

In this appendix, Theorems 1 and 3 are reformulated into the language of cubical homology and proven. For mathematical details, see [10]. The following result is equivalent to Theorem 1.

Theorem 4

Let $X \subset \mathbb{R}^d$ be a full cubical set. Let $\mathcal{U}(C;X) := \{U \in \mathcal{K}_d(X) \mid U \cap C \neq \emptyset\} \setminus \{C\}$, where $\mathcal{K}_d$ denotes the set of all elementary cubes in $\mathbb{R}^d$ and $\mathcal{K}_d(X) := \{Q \in \mathcal{K}_d \ : \ Q \subset X\}$. Let

$$A := \bigcup_{U \in \mathcal{U}(C;X)} U.$$

Then, the point associated with the cube C is a homology-simple point if and only if A is acyclic.

Proof: We begin by showing that the simplicity of the point associated with C is determined by A. Let

$$\mathcal{M} := \mathcal{K}_d(X) \setminus \{C\} \quad \text{and} \quad \mathcal{N}(C) = \{C\} \cup \mathcal{U}(C;X).$$

Set $M = \bigcup_{U \in \mathcal{M}} U$ and $N = \bigcup_{U \in \mathcal{N}} U$. By definition the point associated to C is homology-simple if and only if $H_*(M) \cong H_*(X)$ with this isomorphism being induced by the inclusion map $i : M \to X$.

Observe that $A = M \cap N$ and $X = M \cup N$. The associated Mayer-Vietoris sequence is

$$\cdots \to H_k(A) \xrightarrow{\phi_k} H_k(M) \oplus H_k(N) \xrightarrow{\psi_k} H_k(X) \xrightarrow{\partial_k} H_{k-1}(A) \to \dots \tag{1}$$

However, C is a strong deformation retract of N and hence N is acyclic. Therefore, for $k \geq 1$, (1) reduces to

$$\cdots \to H_k(A) \xrightarrow{\phi_k} H_k(M) \xrightarrow{i_k} H_k(X) \xrightarrow{\partial_k} H_{k-1}(A) \to \dots \tag{2}$$

By definition the point associated to C is simple if and only if i_k is an isomorphism for all k. However, if i_k is an isomorphism, then the images of ϕ_k and ∂_k are trivial. This implies that $H_k(A) = 0$ for $k \geq 1$.

Again, the acyclicity of N implies that for $k=0$ equation (1) reduces to

$$0 \to H_0(A) \stackrel{\phi_0}{\to} H_0(M) \oplus \mathbb{Z} \stackrel{\psi_0}{\to} H_0(X) \stackrel{\partial_0}{\to} 0 \tag{3}$$

Since the zeroth homology group of any space is free, $\ker \psi_0 \cong \mathbb{Z}$ if and only if $H_0(A) \cong \mathbb{Z}$. ∎

We now turn to the proof of Theorem 3. If $d \leq 2$, then the proof is a triviality and can be done by inspection. Thus we present the details in the case $d=3$. Without loss of generality let us assume that $C = [0,1]^3$ is the elementary cube associated with $p \in \mathcal{P} \subset \mathbb{R}^3$. Let $X := CN(p) \cap C$. It is straightforward though technical to check that X is homotopy equivalent to $CN(p)$. Thus, to prove Theorem 3 it is sufficient to prove that we can construct a contraction from C to X.

By Theorem 2 the hypothesis of Theorem 3 is equivalent to the acyclicity of X.

We will make extensive use of the elementary collapses in our construction of a contraction, hence we review some of the essential definitions and properties. Let $\mathcal{Z}$ be a cubical complex. Let $P \in \mathcal{Z}$ be an elementary cube with a free face $Q \subset P$. The elementary collapse of P by Q results in the new cubical complex $\mathcal{Z}' := \mathcal{Z} \setminus \{P, Q\}$. Let Z and Z' denote the cubical sets defined by the cubical complexes $\mathcal{Z}$ and $\mathcal{Z}'$, respectively. Two important facts are that $H_*(Z) \cong H_*(Z')$ and Z' is a deformation retract of Z.

Let $\mathcal{K}(X)$ and $\mathcal{K}(C)$ denote the sets of elementary cubes associated with X and $C = [0,1]^3$. Furthermore, $\mathcal{K}_k(X)$ denotes the set of k-dimensional elementary cubes in X, etc. Our goal is to reduce $\mathcal{K}(C)$ to $\mathcal{K}(X)$ via a series of elementary collapses. To do this we need to introduce some additional notation.

In general given two cubical sets $Y \subset Z$, e.g. $X \subset C$, $Z \setminus Y$ is not a cubical set. However, it can be written as the union of elementary cells [10, Definition 2.13] which we denote by $\mathcal{M}$. Furthermore, let $\mathcal{M}_k$ denote the set of k-dimensional elementary cells in $\mathcal{M}$. Let $\overset{\circ}{P} \in \mathcal{M}_k$. Then $P := \mathrm{cl}(\overset{\circ}{P})$ is a

k-dimensional elementary cube. Let $\overset{\circ}{Q} \in \mathcal{M}_{k-1}$ and $Q := \mathrm{cl}(\overset{\circ}{Q})$. $\overset{\circ}{Q}$ is a *free face of* $\overset{\circ}{P}$ *with respect to* Y, if $\overset{\circ}{P}$ is the unique elementary cell in $\mathcal{M}_k$ such that $Q \subset P$ and $Q \not\subset Y$.

We can now begin our series of elementary collapses. Let $\mathcal{M}^3$ denote the set of elementary cells which are subsets of $C \setminus X$. Observe that $\overset{\circ}{P} = (0,1)^3 \in \mathcal{M}_3^3$. Since $H_2(X) = 0$, $\mathcal{K}_2(X) \neq \mathcal{K}_2(C)$. Without loss of generality we can assume that $[0,1]^2 \times [1] \notin \mathcal{K}_2(X)$ or equivalently that $\overset{\circ}{Q} = (0,1)^2 \times [1] \in \mathcal{M}_2^3$. Let C^2 be the cubical set obtained by the elementary collapse of P by Q. Then $C \sim C^2$ and $X \subset C^2$.

Let $\mathcal{M}^2$ denote the set of elementary cells which are subsets of $C^2 \setminus X$. Observe that $\mathcal{K}_3(C^2) = \emptyset$ and hence $\mathcal{M}_3^2 = \emptyset$. If at this stage $\mathcal{M}_2^2 = \emptyset$, then $C^2 \setminus X = \emptyset$ and we are done. So assume $\mathcal{M}_2^2 \neq \emptyset$. We now proceed to remove elements of $\mathcal{M}_2^2$ via elementary collapses. This involves proving the existence of free faces with respect to X.

Lemma 1

Let $\mathcal{Z} := \{(0,1) \times [\alpha] \times [1], [\alpha] \times (0,1) \times [1] \mid \alpha \in \{0,1\}\}$. *If* $\mathcal{M}_2^2 \neq \emptyset$, *then there exists* $\overset{\circ}{Q} \in \mathcal{M}_1^2 \cap \mathcal{Z}$ *which is a free face for some* $\overset{\circ}{P} \in \mathcal{M}_2^2$.

Proof: If $\mathcal{M}_1^2 \cap \mathcal{Z} = \emptyset$, then

$$\mathcal{Y} := \{[0,1] \times [\alpha] \times [1], [\alpha] \times [0,1] \times [1] \mid \alpha \in \{0,1\}\} \subset \mathcal{K}_1(X).$$

Let $Y := \cup_{Q \in \mathcal{Y}} Q$. Observe that $H_1(Y) \cong \mathbb{Z}$. Furthermore, $Y \subset X$. However, we are assuming that $\mathcal{M}_2^2 \neq \emptyset$ which precludes the assumption that X is acyclic.

Since X is a cubical set, if $\overset{\circ}{Q} = (0,1) \times [\alpha] \times [1] \in \mathcal{M}_1^2$, then $[0,1] \times [\alpha] \times [0,1] \notin \mathcal{K}_2(X)$ and hence $\overset{\circ}{P} = (0,1) \times [\alpha] \times (0,1) \in \mathcal{M}_2^2$, and $\overset{\circ}{Q}$ is a free face of $\overset{\circ}{P}$. The same argument applies to the other elements of $\mathcal{Z}$. ∎

Using Lemma 1 we can perform at least one elementary collapses. Let $C^{2,i}$ denote the resulting space after i such collapses and let $\mathcal{M}^{2,i}$ denote the set of elementary cells which are subsets of $C^{2,i} \setminus X$.

Lemma 2

If $\mathcal{M}_2^{2,i} \neq \emptyset$*, then there exists* $\overset{\circ}{Q} \in \mathcal{M}_1^{2,i}$ *which is a free face relative to* X *for some* $\overset{\circ}{P} \in \mathcal{M}_2^{2,i}$*.*

Proof: Let

$$M^{2,i} := \bigcup \left\{ Q \mid \overset{\circ}{Q} \in \mathcal{M}^{2,i} \right\}$$

Since $i \geq 1$, we can without loss of generality assume that $(0,1)^2 \times [1], [1] \times (0,1)^2 \notin \mathcal{M}_2^{2,i}$. This implies that $H_1(M^{2,i}) = 0$.

Let $\overset{\circ}{P} \in \mathcal{M}_2^{2,i}$. If $Q \subset P$ and $\dim Q = 1$, then $\overset{\circ}{Q}$ can fail to be a free face of $\overset{\circ}{P}$ relative to X in two ways. Either $Q \subset X$ or $\overset{\circ}{Q} \in \mathcal{M}_1^{2,i}$ and $Q \subset P' \neq P$ where $\overset{\circ}{P'} \in \mathcal{M}_2^{2,i}$.

We now proceed using a proof by contradiction. Observe that if X is acyclic using $\mathbb{Z}$ coefficients, then it is acyclic using $\mathbb{Z}_2$ coefficients. Let $\mathcal{M}_2^{2,i} = \{\overset{\circ}{P}_j \mid j = 1, \ldots, J\}$. Consider the chain

$$z := \sum_{j=1}^{J} P_j$$

where we have identified the elementary cube with a basis element of $C_2(C^{2,i}; \mathbb{Z}_2)$. Observe that $\partial z \in C_1(M^{2,i} \cap X; \mathbb{Z}_2)$ and hence $\partial z \in Z_1(M^{2,i} \cap X; \mathbb{Z}_2)$. Since $\mathcal{M}_2^{2,i} \cap \mathcal{K}_2(X) = \emptyset$, ∂z generates a nontrivial element of $H_1(M^{2,i} \cap X; \mathbb{Z}_2)$. Now consider the Mayer-Vietoris sequence for $C^{2,i} = M^{2,i} \cup X$.

$$\to H_2(C^{2,i}; \mathbb{Z}_2) \to H_1(M^{2,i} \cap X; \mathbb{Z}_2) \to H_1(M^{2,i}; \mathbb{Z}_2) \oplus H_1(X; \mathbb{Z}_2) \to$$

$C^{2,i}$ was obtained by a sequence of deformation retracts from $[0,1]^3$ and hence is acylic. Thus, we the following exact sequence

$$\to 0 \to \mathbb{Z}_2^{\lambda} \to 0 \oplus 0 \to$$

where $\lambda \geq 1$. Clearly a contradiction. ∎

Using Lemma 2 we can continue to perform elementary collapses until $\mathcal{M}_2^{2,i} = \emptyset$. Let $C^1 := C^{2,i}$. Let $\mathcal{M}^1$ denote the set of elementary cells which are subsets of $C^1 \setminus X$. If $\mathcal{M}_1^1 = \emptyset$ we are done, i.e. $C^1 = X$. Thus, we assume $\mathcal{M}_1^1 \neq \emptyset$.

Lemma 3

If $\mathcal{M}_1^{1,i} \neq \emptyset$, then there exists $\overset{\circ}{Q} \in \mathcal{M}_0^{1,i}$ which is a free face relative to X for some $\overset{\circ}{P} \in \mathcal{M}_1^{1,i}$.

Proof: Let $\mathcal{M}_1^{1,i} = \{\overset{\circ}{P}_j \mid j = 1, \ldots, J\}$ and let $M^{1,i} = \cup_{j=1}^{J} P$. Consider any chain

$$z := \sum_{j=1}^{J} \alpha_j P_j, \quad \alpha_j \in \{0, 1\}$$

where we have identified the elementary cube with a basis element of $C_1(C^{1,i}; \mathbb{Z}_2)$.

Assume that $\partial z = 0$ so that $z \in Z_1(C^{1,i}; \mathbb{Z}_2)$. Since $C^{1,i}$ is acyclic, there exists $\{R_1, \ldots, R_L\} \subset \mathcal{K}_2(C^{1,i})$ such that $\partial \sum_{l=1}^{L} R_l = z = \sum_{j=1}^{J} \alpha_j P_j$. However, $\mathcal{M}_2^{1,i} = \emptyset$ hence $\{R_1, \ldots, R_L\} \subset \mathcal{K}_2(X)$ which precludes $P_j \in \mathcal{M}_1^{1,i}$, a contradiction. Therefore, $\partial z \neq 0$.

Assume that $\partial z \in C_0(M^{1,i} \cap X; \mathbb{Z}_2)$. This will be the case if $\overset{\circ}{Q} \in \mathcal{M}_0^{1,i}$ implies that there exist exactly two elementary cells $\overset{\circ}{P}, \overset{\circ}{P'} \in \mathcal{M}_1^{1,i}$ such that $P \cap P' = Q$. Since X is acyclic, there exists $\{R_1, \ldots, R_L\} \subset \mathcal{K}_1(X)$ such that $\partial \sum_{l=1}^{L} R_l = \partial z$. Now observe that $\partial(z + \sum_{l=1}^{L} R_l) = 0$ and hence $z + \sum_{l=1}^{L} R_l \in Z_1(C^{1,i}; \mathbb{Z}_2)$. Since $C^{1,i}$ is acyclic, there exist $\{T_1, \ldots, T_N\} \subset \mathcal{K}_2(C^{1,i})$ such that $\partial \sum_{n=1}^{N} T_n = z + \sum_{l=1}^{L} R_l$. Since $z := \sum_{j=1}^{J} P_j$ and $\overset{\circ}{P}_j \in \mathcal{M}_1^{1,i}$, not all T_n can be elements of $\mathcal{K}_2(X)$, thus there exists $T_n \in \mathcal{M}_2^{1,i}$, a contradiction.

Let $z = \sum_{j=1}^{J} P_j$. Given the preceding arguments, $\partial z \neq 0$ and $\partial z \notin C_0(M^{1,i} \cap X; \mathbb{Z}_2)$. This implies that z is a tree, i.e a graph composed of vertices and edges with no cycles. Moreover, assume that $\mathcal{M}_0^{1,i}$ has no free faces relative to X. Since z is a tree, it must have at least two vertices which are free faces, and hence these vertices must be in X.

Let us denote two of these free vertices by L_1 and L_2. We may assume that there is a path in the tree with edges $\{P_{k_1}, \ldots P_{k_K}\} \subset \mathcal{M}_1^{1,i}$ from L_1 to L_2. However, since X is acyclic, L_1 and L_2 are connected by edges $\{R_1, \ldots, R_L\} \subset \mathcal{K}_1(X)$. Now observe that $\partial(\sum_{i=1}^{k} P_{k_i} + \sum_{l=1}^{L} R_l) = 0$ and we have a contradiction as described above. ∎

Using Lemma 3 we can continue to perform elementary collapses until $\mathcal{M}_1^{1,i} = \emptyset$. Let $C^0 := C^{1,i}$. Let $\mathcal{M}^0$ denote the set of elementary cells which are subsets of $C^0 \setminus X$. Since C^0 is acyclic, it is connected. Therefore, $\mathcal{M}^0 = \emptyset$ and the proof is completed.

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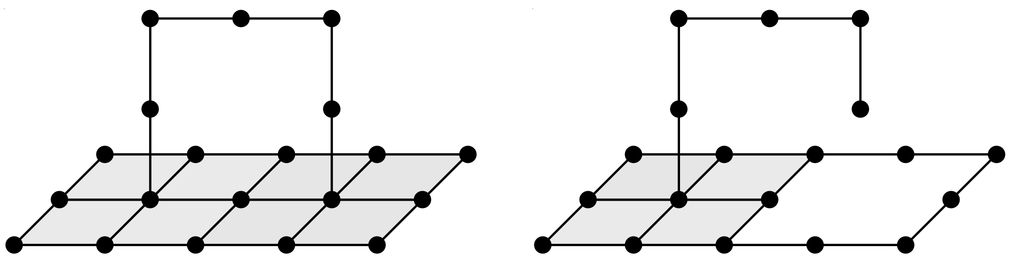
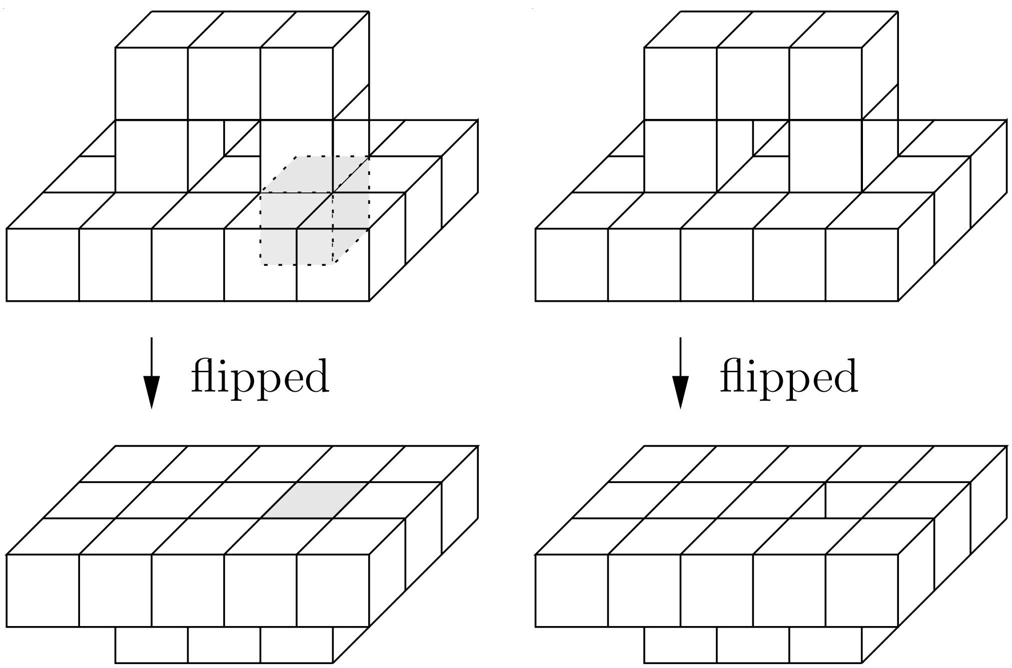


(a) Vertical loop. (b) Horizontal loop.



(c) Vertical loop represented by cubes. (d) Removing the shaded cube preserves the vertical loop.

Fig. 7. $(2d)$-connectivity between points (top of the figure) allows for a change in topology when a point is removed even though the homology groups are isomorphic. Removing a point with an acyclic cubical neighborhood using $(3^d - 1)$ connectivity does not allow such a change in topology.