Sprague-Grundy Functions for Divisor Graphs

Aaron Meyerowitz, Florida Atlantic University

Consider these graphs: $\mathbb{N}^{\downarrow}$ has as vertices the positive integers and an edge $q \to q-d$ for every *proper* divisor $d|q$, while $\mathbb{N}^{\uparrow}$ has the same vertices and edges, but with direction reversed. So an edge $p \to p+d$ for every divisor $d|p$. The first, $\mathbb{N}^{\downarrow}$, could be considered a terminating take-away game and has a unique Sprague-Grundy function (SGf), call it g. That function, which we determine, is also a SGf for $\mathbb{N}^{\uparrow}$, but not by far not the only one. If we restrict $\mathbb{N}^{\downarrow}$ to an interval $[1, T]$ we get the same SGf. The SGf, g_T, for the restriction of $\mathbb{N}^{\uparrow}$ to $[1, T]$, is only partially understood. But there seems to be strong evidence that

$$\lim_{T\to\infty} g_T = g$$

in the sense that $g_T(n) = g(n)$ for all large enough T.

Keywords: number theory, combinatorial games.