A skeptical perspective on the 2n Conjecture

Tracy Hall, NewVistas LLC (Provo, UT)

An $n \times n$ matrix over some field $\mathbb{F}$ specifies a *pattern* of entries from $\{0, *\}$ (zero and nonzero). If $\mathbb{F}$ is ordered, it also specifies a *sign pattern* with entries in $\{0, +, -\}$. A pattern or sign pattern is called *spectrally arbitrary* over $\mathbb{F}$ if every monic polynomial of degree n is the characteristic polynomial of some matrix in the class. It is known that this requires at least $2n-1$ nonzero entries, and it is widely expected (the *2n Conjecture*) that in fact $2n$ nonzero entries are required.

The conjecture has in its favor that exhaustive search has shown it to hold for small n, and that the main (and perhaps only known) tool for proving that a pattern is spectrally arbitrary, the Nilpotent-Jacobian method, fails with only $2n - 1$ nonzero entries. Although these facts explain why no counterexamples are known, I will argue that they should not be taken as convincing evidence that no counterexamples exist.

The main new contribution is an invariant defined on any strongly connected digraph Γ on n vertices with exactly $2n - 1$ directed edges and loops. The invariant takes the form of a nonnegative integer $d_\sigma(\Gamma)$ called the *spectral covering degree* of Γ. The parametrized family of $n \times n$ matrices with pattern given by Γ maps by way of the characteristic polynomial to a coefficient vector in $\mathbb{F}^n$, and the size of $d_\sigma(\Gamma)$ is a measure of how much flexibility one has when trying to invert this mapping. In the absence of a combinatorial definition, the invariant becomes cumbersome to calculate for large n, but calculations of $d_\sigma(\Gamma)$ for moderate values of n have yielded some large (and surprising) values. This seems to hint that as n increases the invariant may often take values so large that one would expect to find every spectrum, even with only $2n - 1$ nonzero entries, for some patterns over $\mathbb{C}$, some patterns over $\mathbb{R}$, or even some sign patterns over $\mathbb{R}$.

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