The bipartite graphs $D'(k,q)$ have low girth

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In [**M.**, Eur. J. Comb. 118, May 2024], we introduced a new family of 3-graphs $D_3(k,q)$, extending the Lazebnik-Ustimenko-Woldar graphs $D(k,q)$ for $k \geq 1$ and prime power q. We noted that when $3 \mid q$, the link graphs of $D_3(k,q)$ were either $D(k,q)$ or a new family of graphs denoted by $D'(k,q)$. Let $\widetilde{C}_{2k}$ denote the hypergraph suspension of a $2k$-cycle, obtained by adding a common new vertex to every edge of C_{2k}. Our work then implied that $\mathrm{ex}_3(n, \widetilde{C}_6) = \Theta(n^{7/3})$ and $\mathrm{ex}_3(n, \widetilde{C}_8) = \Omega(n^{11/5})$. We noted that a lower bound of $k+5$ for odd k on the girth of $D'(k,q)$ for $q = 3^r$ would prove the lower bound $\mathrm{ex}_3(n, \widetilde{C}_{2k}) \geq \Omega(n^{2+\frac{1}{2k-3}})$ on the 3-graph Turán number of $\widetilde{C}_{2k}$.

In this work, we will show that $D'(k,q)$ and $D(k,q)$ have the same number of connected components when $q = 3^r$. However, for $k \geq 5$ and $q \geq 4$, $D'(k,q)$ contains cycles of length 10. This suggests the need to define hypergraph families other than $D_3(k,q)$ to obtain lower bounds on $\mathrm{ex}_3(n, \widetilde{C}_{2k})$ for $k \geq 5$. Also, it remains unknown whether the asymptotics of $\mathrm{ex}_3(n, \widetilde{C}_8)$ is $\Theta(n^{11/5})$ or $\Theta(n^{9/4})$.

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