Rational Exponents for Cliques

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The extremal number of the graph F, denoted $\mathrm{ex}(n, F)$, is the maximum number of edges in an F-free graph on n vertices. The inverse rational exponent conjecture (perhaps first posed by Erdős and Simonovits in '81) postulates that for each rational number $r \in [1, 2]$, there exists some graph F such that

$$\mathrm{ex}(n, F) = \Theta(n^r).$$

Recently, Bukh and Conlon proved a slightly weaker version of this conjecture - if one allows for finite families of forbidden graphs, then such a family does exist for each rational r.
We will show that a generalization of this conjecture also holds. Given two graphs F and H, the generalized extremal number $\mathrm{ex}(n, H, F)$ is the maximum number of copies of H in an F-free graph on n vertices (note that $\mathrm{ex}(n, F) = \mathrm{ex}(n, K_2, F)$). We will explore which rational exponents are realizable for some different graphs H, and in particular show that every reasonable rational number is realizable for all cliques K_ℓ. Upper bounds will be derived from a particular counting scheme, while lower bounds will stem from a random algebraic construction.