Quiz 3

1.

(a) How many phone numbers are there of the form

"561–XXX–XXXX"?

Since there are 10 digits from 0 to 9, and because repetition is allowed, we have 10 choices for the first X in the phone number, 10 choices for the second X, and so on, and 10 choices for the 7th X of the number. There are $10^7 = 10,000,000$ possible phone numbers.

(b) License plates consist of six characters: The first three characters are uppercase letters (A - Z), and the last three characters are digits (0 - 9). How many license plates are possible if no character may be repeated on the same plate?

Hint: There are 26 letters to the English alphabet.

There are 26 ways to select the first character of the plate. Once this is done, there are 25 choices for the second character, and there are 24 ways to fill third character, because repetition is not allowed. For the first three characters there are $(26)_3 = 26 \cdot 25 \cdot 24$ different license plates. For each of them we have 10 choices for the 4th character (because there are 10 digits from 0 to 9). For each of them we have 9 choices for the 5th character. And for each of them we have 8 choices for the 6th character. Therefore we have $(26)_3 \cdot (10)_3 = 26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8$ possible license plates.
(c) A bookshelf contains 4 blue books and 2 green books. In how many different orders can these books be arranged on the shelf if the green books are standing next to one another?

There are \((4)_4 = 1 \cdot 2 \cdot 3 \cdot 4 = 24\) ways to arrange four blue books on a bookshelf, namely \(\_ \_ \_ \_\) is one of

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 2 & 4 & 3 \\
1 & 3 & 2 & 4 \\
1 & 3 & 4 & 2 \\
1 & 4 & 2 & 3 \\
1 & 4 & 3 & 2 \\
\end{array}
\]

Two green books are placed next to one another in one of the following ways:

\[
\begin{array}{cccc}
1 & 2 & \_ & \_ \\
\_ & 1 & 2 & \_ \\
\_ & \_ & 1 & 2 \\
\_ & \_ & \_ & 1 \\
\_ & \_ & \_ & \_ \\
\end{array}
\]

There are \((4)_4 \cdot 2 \cdot 5 = 24 \cdot 10 = 240\) different orders to arrange the books on the shelf.
2.

(1) Here is another way to write $5 \cdot 7 \cdot 9 \cdot 11 \cdot 13$, using the product symbol $\prod$:

$$5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 = \prod_{k=1}^{5} (2 \cdot k + 3).$$

Using the product symbol express

(a) $$(5)_3 = 5 \cdot 4 \cdot 3 = \prod_{k=1}^{3} (5 - k + 1)$$

(b) $$(n)_k = n \cdot (n - 1) \cdots (n - k + 1) = \prod_{t=1}^{k} (n - t + 1)$$

(2) For what values of $n$ and $k$ is the following equation correct?

$$n! = (n)_k \cdot (n - k)!$$

**Case 1.** For $0 \leq n < k$ on the left side of the equation $n! = (n)_k \cdot (n - k)!$ we have $n! \neq 0$ and on the right side $(n)_k = n \cdot (n - 1) \cdots (n - n) \cdot (n - n - 1) \cdots (n - k + 1) = 0$, and $(n - k)!$ is undefined, because $n - k < 0$. Therefore in this case, the equation $n! = (n)_k \cdot (n - k)!$ is not correct.

**Case 2.** For $0 \leq k \leq n$ on the left side of the equation $n! = (n)_k \cdot (n - k)!$ we have

$$n! = 1 \cdot 2 \cdots (n - k - 1) \cdot (n - k) \cdot (n - k + 1) \cdots (n - 1) \cdot n$$

and on the right side

$$(n)_k \cdot (n - k)! = \underbrace{n \cdot (n - 1) \cdots (n - k + 1)}_{(n)_k} \cdot \underbrace{(n - k) \cdot (n - k - 1) \cdots 2 \cdot 1}_{(n - k)!}.$$

In this case, the equation $n! = (n)_k \cdot (n - k)!$ is correct, because multiplication of integers is commutative,
3. Exercise for Extra Credit (5 points)
A bookshelf contains 4 blue books and 3 green books. In how many
different orders can these books be arranged on the shelf if the green
books are NOT standing next to one another?

There are \((3)_3 = 1 \cdot 2 \cdot 3 = 6\) ways for three green books to be
located as follows:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 3 & 2 & 4 \\
2 & 1 & 3 & 4 \\
2 & 3 & 1 & 4 \\
3 & 1 & 2 & 4 \\
3 & 2 & 1 & 4
\end{array}
\]

For each of them there are \((4)_4 = 1 \cdot 2 \cdot 3 \cdot 4 = 24\) ways to
arrange four blue. Therefore there are \((4)_4 \cdot (3)_3 = 24 \cdot 6\) possible orders
for the blue and green books to be in the pattern:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 3 & 2 & 4 \\
2 & 1 & 3 & 4 \\
2 & 3 & 1 & 4 \\
3 & 1 & 2 & 4 \\
3 & 2 & 1 & 4
\end{array}
\]

A similar explanation yields:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 3 & 2 & 4 \\
2 & 1 & 3 & 4 \\
2 & 3 & 1 & 4 \\
3 & 1 & 2 & 4 \\
3 & 2 & 1 & 4
\end{array}
\]

There are \((4)_4 \cdot (3)_3 \cdot 10 = 24 \cdot 6 \cdot 10\) different orders to arrange
the books on the shelf.