Quiz 7

1.

(a) What does the symbol \( \binom{n}{k} \) mean?

The symbol \( \binom{n}{k} \) denotes the number of \( k \)-element subsets (i.e. the subsets of \( k \)-elements) of an \( n \)-element set.

In other words:

\( \binom{n}{k} \) is the number of subsets of \( \{1, 2, \ldots, n\} \) having \( k \) elements.

In other words:

\( \binom{n}{k} \) is the number of subsets of \( \{1, 2, \ldots, n\} \) with cardinality \( k \).

(b) Explain in words why the equation \( \binom{n}{k} = \binom{n}{n-k} \) is true.

When the cardinality of a subset \( U \) of \( \{1, 2, \ldots, n\} \) is \( k \) (i.e. \( |U| = k \), the number of elements in \( U \) is \( k \)), then the cardinality of the subset \( \{1, 2, \ldots, n\} - U \) is \( n - k \) (we remove \( k \) elements from \( n \) elements). - and vice versa: when the cardinality of a subset \( V \) of \( \{1, 2, \ldots, n\} \) is \( n - k \), then the cardinality of the subset \( \{1, 2, \ldots, n\} - V \) is \( n - (n - k) = k \).

That is, each \( k \)-element subset of \( \{1, 2, \ldots, n\} \) corresponds to a unique \( n - k \)-element subset of \( \{1, 2, \ldots, n\} \) which is its complement - and vice versa.

Therefore the number of \( k \)-element subsets of \( \{1, 2, \ldots, n\} \), namely \( \binom{n}{k} \), is the same as the number of \( (n - k) \)-element subsets of \( \{1, 2, \ldots, n\} \), namely \( \binom{n}{n-k} \).
In other words:

The counting of the $k$-element subsets of $\{1, 2, \ldots, n\}$ is the counting of the $(n-k)$-element subsets of $\{1, 2, \ldots, n\}$, namely the compliments of $k$-element sets.

(c) To win the lottery you, must pick six numbers from 49 balls numbered 1-49 (the order in which you pick the balls doesn’t matter). How many different combinations can you pick?

The number of combinations is the number of 6-element subsets of $\{1, 2, \ldots, 49\}$. Thus there are \( \binom{49}{6} = \frac{49!}{6! \cdot 43!} = \frac{44 \cdot 45 \cdot 46 \cdot 47 \cdot 48 \cdot 49}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \) different combinations you can pick.
2.

(a) Place the following binomial coefficients in Pascal’s Triangle:

\[
\binom{4}{1}, \binom{5}{2}, \binom{6}{3}, \binom{7}{3}.
\]

(b) Prove \( \binom{7}{3} = \binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3} \).

Using \( \binom{n + 1}{k + 1} = \binom{n}{k} + \binom{n}{k + 1} \) (Pascal’s Identity) we have:

\[
\binom{7}{3} = \binom{6}{2} + \binom{6}{3} = \frac{\binom{6}{2} + \binom{6}{3}}{\binom{1}{1} + \binom{2}{1}} = \binom{5}{1} + \binom{5}{2} + \binom{6}{3} = \binom{4}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3}.
\]

Since \( \binom{3}{0} = \binom{4}{0} = 1 \) (the subset \( \emptyset \) is the only subset of \( \{1, 2, 3\} \) having 0-elements, i.e. \( \binom{3}{0} = 1 \), and (the subset \( \emptyset \) is the only subset of \( \{1, 2, 3, 4\} \) having 0-elements, i.e. \( \binom{4}{0} = 1 \) ), we have

\[
\binom{7}{3} = \binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3}.
\]
(c) Generalize the equation (b) for every \( n \in \mathbb{N} \)

- \( n = 0 \). We have \( \binom{0}{0} = \binom{3}{0} = 1 \).
- \( n = 1 \). We have \( \binom{1}{1} = \binom{3}{0} + \binom{4}{1} = \binom{4}{0} + \binom{4}{1} \).
- \( n = 2 \). We have \( \binom{2}{1} + \binom{5}{2} = \binom{3}{0} + \binom{4}{1} + \binom{5}{2} \).
- \( n = 3 \). We have \( \binom{3}{2} = \binom{6}{3} + \binom{6}{2} = \binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3} \).
- \( n = 4 \). We have \( \binom{4}{3} = \binom{7}{4} + \binom{5}{4} = \binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3} + \binom{7}{4} \).
- \( n = 5 \). We have \( \binom{5}{4} = \binom{6}{5} + \binom{6}{4} + \binom{7}{5} + \binom{8}{5} \).

\[
\begin{array}{cccccccc}
\binom{1}{0} & \binom{1}{1} \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \\
\binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} \\
\binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\binom{n+1}{0} & \binom{n+1}{1} & \binom{n+1}{2} & \ldots & \binom{n+1}{n-2} & \binom{n+1}{n-1} & \binom{n+1}{n} \\
\binom{n+2}{0} & \binom{n+2}{1} & \binom{n+2}{2} & \ldots & \binom{n+2}{n-2} & \binom{n+2}{n-1} & \binom{n+2}{n} \\
\binom{n+3}{0} & \binom{n+3}{1} & \binom{n+3}{2} & \ldots & \binom{n+3}{n-2} & \binom{n+3}{n-1} & \binom{n+3}{n} \\
\binom{n+4}{0} & \binom{n+4}{1} & \binom{n+4}{2} & \ldots & \binom{n+4}{n-2} & \binom{n+4}{n-1} & \binom{n+4}{n} \\
\end{array}
\]

\[
\binom{n+4}{n} = \binom{3}{0} + \binom{4}{1} + \ldots + \binom{i+3}{i} + \ldots + \binom{n+3}{n} \\
= \sum_{i=0}^{n} \binom{i+3}{i}
\]
3. Exercise for Extra Credit (5 points)

(a) What is the coefficient of \( x^3y^3 \) in \((x + y)^6\). 

\[ \binom{3 + 3}{3} = \binom{6}{3} \]

(b) Calculate \( \binom{6}{3} \).

\[ \binom{6}{3} = \frac{6!}{3! \cdot (6 - 3)!} = \frac{4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3} = 20 \]

(c) The number \( \binom{26}{19} \) is the coefficient that appears in the expansion of

- \((x + y)^{26}\)
- as coefficient of \(x^{26-19}y^{19} = x^7y^{19}\).