1. For each of the following congruences, find two integers \( x \), that make the congruence true.

- \( 1 \equiv x \pmod{3} \).
  
  For example
  - \( x = -32 \), because \( 1 - (-32) = 33 = 3 \cdot 11 \) is divisible by 3.
  - \( x = 10 \), because \( 1 - 10 = -9 = 3 \cdot (-3) \) is divisible by 3.

- \( x \equiv 1 \pmod{3} \).
  
  For example
  - \( x = -32 \), because \( -32 - 1 = -33 = 3 \cdot (-11) \) is divisible by 3.
  - \( x = 10 \), because \( 10 - 1 = 9 = 3 \cdot 3 \) is divisible by 3.

- \( x \equiv x \pmod{3} \).
  
  For example
  - \( x = -32 \), because \( -32 - (-32) = 0 = 3 \cdot 0 \) is divisible by 3.
  - \( x = 10 \), because \( 10 - 10 = 0 = 3 \cdot 0 \) is divisible by 3.

(b) Prove that the relation \( R = \{ (x, y) : x \equiv y \pmod{3} \} = \{ (x, y) : 3 \mid (x - y) \} \) on \( \mathbb{Z} \) is an equivalence relation.

(a) The relation \( R = \{ (x, y) : 3 \mid (x - y) \} \) is reflexive.

(We have to show that for any integer \( x \in \mathbb{Z} \) we have \( (x, x) \in R \))

Proof. Let \( x \in \mathbb{Z} \). Since \( 3 \mid 0 \) and \( x - x = 0 \), we have \( 3 \mid (x - x) \). Therefore \( (x, x) \in R \). Since \( x \) is arbitrarily selected, we have shown \( (x, x) \in R \) for all \( x \in \mathbb{Z} \).

(b) The relation \( R = \{ (x, y) : 3 \mid (x - y) \} \) is symmetric.

(We have to show that if \( (m, n) \in R \), then \( (n, m) \in R \))

Proof. Assume \( (m, n) \in R \), then \( 3 \mid (m - n) \). By the definition of "divisible" there exists an integer \( a \) such that \( m - n = 3a \). Since \( n - m = -m + n = -(m - n) = -3a = 3(-a) \), we obtain that \( 3 \) divides \( n - m \). Therefore \( (n, m) \in R \).

(c) The relation \( R = \{ (x, y) : 3 \mid (x - y) \} \) is transitive.

(We have to show that if \( (m, n) \) and \( (n, r) \) are in \( R \), then \( (m, r) \in R \))

Proof. Assume \( (m, n), (n, r) \in R \), then there exist \( a, b \in \mathbb{Z} \), such that \( m - n = 3a \) and \( n - r = 3b \). This can be rewritten as
\[ m = n + 3a \quad \text{and} \quad r = n - 3b. \]

Since
\[ m - r = (n + 3a) - (n - 3b) = 3a + 3b = 3(a + b), \]
we obtain that \( m - r \) is divisible by 3, i.e. \( 3 | (m - r) \). Therefore \( (m, r) \in R \).
2. Let $a \in \mathbb{Z}$. What is an equivalence class $[a]$ of $a$ related to the equivalence relation $R = \{(x,y) : x \equiv y \pmod{3}\}$ on $\mathbb{Z}$?

\[ [a] = \{ x \in \mathbb{Z} : (a,x) \in R \} \]
\[ = \{ x \in \mathbb{Z} : a \equiv y \pmod{3} \} \]
\[ = \{ x \in \mathbb{Z} : 3|(a-y) \} \]
\[ = \{ x \in \mathbb{Z} : a-x = 3n, \ n \in \mathbb{Z} \} \]
\[ = \{ x \in \mathbb{Z} : x = a-3n, \ n \in \mathbb{Z} \} \]

(a) Find the requested equivalence classes

- $[0] = \{ x \in \mathbb{Z} : x = 0-3n = 3(-n), \ n \in \mathbb{Z} \}$
  \[ = \{ \cdots -9, -6, -3, 0, 3, 6, 9, \ldots \} \]

- $[1] = \{ x \in \mathbb{Z} : x = 1-3n, \ n \in \mathbb{Z} \}$
  \[ = \{ \cdots -8, -5, -2, 1, 4, 7, 10, \ldots \} \]

- $[2] = \{ x \in \mathbb{Z} : x = 2-3n, \ n \in \mathbb{Z} \}$
  \[ = \{ \cdots -7, -4, -1, 2, 5, 8, 11, \ldots \} \]

- $[3] = \{ x \in \mathbb{Z} : x = 3-3b = 3(1-n), \ n \in \mathbb{Z} \}$
  \[ = \{ \cdots -9, -6, -3, 0, 3, 6, 9, \ldots \} \]

(b) Determine the number of equivalence classes of $R$. Verify your answer.

Let $S$ be the set of all equivalence classes with respect to $R$ i.e.

\[ S = \{ [z] : z \in \mathbb{Z} \} = \{ \ldots, [-4], [-3], [-2], [-1], [0], [1], [2], [3], \ldots \} \]

then the cardinality of $S$ is 3. More precisely $S = \{ [0], [1], [2] \}$.

Let $z \in \mathbb{Z}$, then either $z = 3a$, or $z = 3a + 1$, or $z = 3a + 2$
where $a \in \mathbb{Z}$.

(For example $31 = 3 \cdot 10 + 1$, $32 = 3 \cdot 10 + 2$, $33 = 3 \cdot 11$, $34 = 3 \cdot 11 + 1$, $35 = 3 \cdot 11 + 2$, $36 = 3 \cdot 12$, etc.)

Case 1. $z = 3a$, then

\[ [3a] = \{ x \in \mathbb{Z} : (3a,x) \in R \} \]
\[ = \{ x \in \mathbb{Z} : 3a-x = 3n, \ n \in \mathbb{Z} \} \]
\[ = \{ x \in \mathbb{Z} : x = 3a+3(-n), \ n \in \mathbb{Z} \} \]
\[ = \{ x \in \mathbb{Z} : x = 3(a-n), \ n \in \mathbb{Z} \} \]
\[ [0] = \{ \ldots, -9, -6, -3, 0, 3, 6, 9, 12, \ldots \} \]
Case 2. \( z = 3a + 1 \), then

\[
[3a + 1] = \{x \in \mathbb{Z} : (3a + 1, x) \in R\} = \{x \in \mathbb{Z} : 3a + 1 - x = 3n, \ n \in \mathbb{Z}\} = \{x \in \mathbb{Z} : x = 3a + 3(-n), \ n \in \mathbb{Z}\} = \{x \in \mathbb{Z} : x = 1 + 3(a - n), \ n \in \mathbb{Z}\}
\]

\[ [1] = \{\ldots, -8, -5, -2, 1, 4, 7, 10, 13, \ldots \} \]

Case 3. \( z = 3a + 2 \), then

\[
[3a + 2] = \{x \in \mathbb{Z} : (3a + 2, x) \in R\} = \{x \in \mathbb{Z} : 3a + 2 - x = 3n, \ n \in \mathbb{Z}\} = \{x \in \mathbb{Z} : x = 3a + 2 + 3(-n), \ n \in \mathbb{Z}\} = \{x \in \mathbb{Z} : x = 2 + 3(a - n), \ n \in \mathbb{Z}\}
\]

\[ [2] = \{\ldots, -7, -4, -1, 2, 5, 8, 11, 14, \ldots \} \]

Since for \( z \in \mathbb{Z} \) we have either \( [z] = [0] \), or \( [z] = [1] \) or \( [z] = [2] \), there are precisely three pairwise different elements in

\[ S = \{[0], [1], [2]\} \]
3. Exercise for Extra Credit (5 points)

(a) Let $A$ be a set. What is a partition on $A$?

A partition of a set $A$ is a set $S$ of subsets of $A$ such that:

1. $\emptyset \not\in S$,
2. the subsets in $S$ are pairwise disjoint,
3. the union of the subsets in $S$ is $A$.

(b) For the given set $A$, determine whether $P$ is a partition of $A$:

(a) $A = \{1, 2, 3, 4, 5, 6, 7\}$, and $P = \\{\{1, 3\}, \{5, 6\}, \{2, 4\}, \{7\}\}$

Yes, because

1. $\emptyset \not\in P$,
2. $\{1, 3\} \cap \{5, 6\} = \emptyset$,
   $\{1, 3\} \cap \{2, 4\} = \emptyset$,
   $\{1, 3\} \cap \{7\} = \emptyset$,
   $\{5, 6\} \cap \{2, 4\} = \emptyset$,
   $\{5, 6\} \cap \{7\} = \emptyset$,
3. $\{1, 3\} \cup \{5, 6\} \cup \{2, 4\} \cup \{7\} = \{1, 2, 3, 4, 5, 6, 7\}$

(b) $A = \mathbb{Z}$, and $P = \\{E, O\}$, where $E$ is the set of all even integers, and $O$ is the set of all odd integers.

Yes, because

1. $\emptyset \not\in P$,
2. $E \cap O = \emptyset$,
3. $E \cup O = \mathbb{Z}$

(c) $A = \mathbb{Z}$ and $P$ is the set of equivalence classes of the equivalence relation $R = \{(x,y) : x \equiv y \pmod{3}\}$.

Yes.

The set $P = \{[0], [1], [2]\}$ is a set of subsets of $\mathbb{Z}$,
(1) $\emptyset \not\in P$, because $[0] \neq \emptyset$, $[1] \neq \emptyset$, $[2] \neq \emptyset$.

(2) $[0] \cap [1] = \emptyset$, $[0] \cap [2] = \emptyset$, and $[1] \cap [2] = \emptyset$. Thus the sets in $P$ are pairwise disjoint.

(3) $[0] \cup [1] \cup [2] = \mathbb{Z}$, because for each $z \in \mathbb{Z}$, we have $z \in [0]$, or $z \in [1]$, or $z \in [2]$.

$P = \{[0], [1], [2]\}$ is a partition on $\mathbb{Z}$. 