Greatest Common Divisor

For any two integers \( a, b \) with \( b > 0 \) there is only one pair \((q,r)\) such that

- \( a = qb + r \)
- \( 0 \leq r < b \)

1. Let \( a = 30 \) and \( b = 13 \). Since \( a = 30 > 0 \) and \( \frac{30}{13} = 2.30769230769\ldots \) we have \( q = 2 \), and \( r = 30 - 2 \cdot 13 = 4 \). Thus for \( q = 2 \) and \( r = 4 \) we have

- \( 30 = \underbrace{2}_{q} \cdot \underbrace{13 + 4}_{r} \)
- \( 0 \leq \underbrace{4}_{r} < \underbrace{13}_{r} \)

**Statement**: For any two integers \( x \) and \( y \) with the properties

- \( 30 = x \cdot 13 + y \) and
- \( 0 \leq y < 13 \),

we have \( x = 2 \) and \( y = 4 \).

**Proof**. We have the fact \( 30 = 2 \cdot 13 + 4 \) and the assumption \( 30 = x \cdot 13 + y \)

\[
\implies \underbrace{2 \cdot 13 + 4}_{30} = \underbrace{x \cdot 13 + y}_{30}
\]

\[
\implies 2 \cdot 13 - x \cdot 13 = y - 4 \cdot 13
\]

\[
\implies (2 - x)13 = y - 4
\]
Since $0 \leq y < 13$, we have $y \in \{0, 1, 2, \ldots, 12\}$, therefore
$y-4 \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$

The last equation $(2-x)13 = y-4$ implies that $y-4$ is a multiple of 13. There is only one integer in $\{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$ that is a multiple of 13, namely 0. Thus $y-4 = 0$ implies $y = 4$.

(We already have proved one part of our statement.)

Moreover, the equation $(2-x)13 = y-4 = 0$ implies $(2-x) = 0$ (because there is only one integer $x$ with the property $x \cdot 13 = 0$, this is $x = 0$). Therefore $x = 2$.

This was what we had to prove.

2. Let $a = -30$ and $b = 12$. Since $a = -30 < 0$, $12 \not{|} (-30)$

\[
\frac{-30}{12} = -2.\text{remainder}\]

implies $q = -2 - 1 = -3$, and $r = 6$. Thus for $q = -3$ and $r = 6$ we have

- $-30 = (-3) \cdot 12 + 6$
- $0 \leq 6 < 12$

**Statement:** For any two integers $x$ and $y$ with the properties

- $-30 = x \cdot 12 + y$
- $0 \leq y < 12$

we have $x = -3$ and $y = 6$.

Proof. We have the fact $-30 = (-3) \cdot 12 + 6$ and the assumption

$-30 = x \cdot 12 + y$

$\Longrightarrow (-3) \cdot 12 + 4 = x \cdot 12 + y$

$\Longrightarrow (-3) \cdot 12 - x \cdot 12 = y - 6$

$\Longrightarrow (-3 - x)12 = y - 6$

Since $0 \leq y < 12$, we have $y \in \{0, 1, 2, \ldots, 11\}$, therefore

$y - 6 \in \{0-6, 1-6, \ldots, 11-6\} = \{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$

The equation $(-3-x)12 = y - 6$ implies that $y - 6$ is a multiple of 12.
There is only one integer in \{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\} that is a multiple of 12, namely 0. Thus \(y - 6 = 0\) implies \(y = 6\). (We already have proved one part of our statement.)

Moreover, the equation \((-3 - x) \cdot 12 = y - 6 = 0\) implies \((-3 - x) = 0\) (because there is only one integer \(x\) with the property \(x \cdot 12 = 0\), this is \(x = 0\)). Therefore \(x = -3\).

This was what we had to prove.

3. **Proposition.** Let \(a\) and \(b\) be positive integers and \(c = a \mod b\) (i.e. \(a = qb + c\) and \(0 \leq c < b\)). Then \(\gcd(a, b) = \gcd(b, c)\)

Apply the Proposition for \(a = 431\) and \(b = 29\):

\[
\begin{align*}
431 & = 14 \cdot 29 + 25 \implies \gcd(431, 29) = \gcd(29, 25) \\
29 & = 1 \cdot 25 + 4 \implies \gcd(29, 25) = \gcd(25, 4) \\
25 & = 6 \cdot 4 + 1 \implies \gcd(25, 4) = \gcd(4, 1) \\
4 & = 4 \cdot 1 + 0 \implies \gcd(4, 1) = \gcd(1, 0) = 1.
\end{align*}
\]

We have

\[
\begin{align*}
\gcd(431, 29) &= \gcd(29, 25) \\
&= \gcd(25, 4) \\
&= \gcd(4, 1) \\
&= \gcd(1, 0) = 1
\end{align*}
\]

We will repeatedly make use of the Proposition above:
1739 = 0 \cdot 29341 + 1739 \implies \gcd(1739, 29341) = \gcd(29341, 1739)

29341 = 16 \cdot 1739 + 1517 \implies \gcd(29341, 1739) = \gcd(1739, 1517)

1739 = 1 \cdot 1517 + 222 \implies \gcd(1739, 1517) = \gcd(1739, 222)

1739 = 7 \cdot 222 + 185 \implies \gcd(1739, 1517) = \gcd(222, 185)

222 = 1 \cdot 185 + 37 \implies \gcd(222, 185) = \gcd(185, 37)

185 = 5 \cdot 37 + 0 \implies \gcd(185, 37) = \gcd(37, 0) = 37

Therefore

\gcd(1739, 29341) = 37

36. 1

a. \gcd(20, 25) = 5

b. \gcd(0, 10) = 10

c. \gcd(123, -123) = 123

d. \gcd(-89, 98) = 1

e. \gcd(54321, 50) = 1

f. \gcd(1739, 29341) = 37