Dividing

For any two integers \( a, b \) with \( b > 0 \) there is only one pair \((q, r)\) such that

\[
\begin{align*}
\bullet & \quad a = q \cdot b + r \\
\bullet & \quad 0 \leq r < b
\end{align*}
\]

A way to determine \( q, r \) for given \( a \) and \( b \): If \( b \mid a \), then \( a = \frac{a}{b} \cdot b \) for some \( \frac{a}{b} \in \mathbb{Z} \). In this case \( q = \frac{a}{b} \) and \( r = 0 \).

We consider the case where \( b \) do not divides \( a \). Note that \( a \) can be positive, or 0, or negative (\( b \) is always positive).

**Case I.** \( a \geq 0 \) (Example: \( a = 23 \) and \( b = 5 \).) In this case we have that the fraction \( \frac{a}{b} \) can be written as a decimal, i.e.

\[
\frac{a}{b} = (\text{integer part of } \frac{a}{b}).(\text{fractional part of } \frac{a}{b})
\]

(Example: \( \frac{23}{5} = 4.6 \), here (integer part of \( \frac{23}{5} \)) is 4, and (fractional part of \( \frac{23}{5} \)) is 6)

In this case

\[
q = (\text{integer part of } \frac{a}{b}) \quad \text{and} \quad r = a - q \cdot b
\]

(Example: \( \frac{23}{5} = 4.6 = 4 \cdot \frac{5}{b} + \frac{3}{r} \))

(Example: \( a = 24 \) and \( b = 32 \), then \( \frac{24}{32} = 0.75 \), thus \( q = 0 \) and \( r = 24 - 0 \cdot 32 = 24 \). Therefore \( \frac{24}{a} = \frac{0}{q} \cdot \frac{32}{b} + \frac{24}{r} \))

**Case II.** \( a < 0 \) (Example: \( a = -23 \) and \( b = 5 \)) In this case

\[
q = (\text{integer part of } \frac{a}{b}) - 1 \quad \text{and} \quad r = a - q \cdot b
\]
(Example: \( \frac{-23}{5} = -4.6 \), here (integer part of \( \frac{23}{4} \)) is \(-4\). Therefore \( q = -4 - 1 = -5 \), thus \( \overline{23} = \overline{5} \cdot \overline{5} + \overline{2} \).)

35.1

a. \( q = 33 \) and \( r = 1 \), because

\[ 100 = \overline{33} \cdot \overline{3} + \overline{1} \]

\[ 0 \leq \frac{1}{r} < \frac{3}{b} \]

b. \( q = -34 \) and \( r = 2 \), because

\[ -100 = \overline{-34} \cdot \overline{3} + \overline{2} \]

\[ 0 \leq \frac{2}{r} < \frac{3}{b} \]

c. \( q = 33 \) and \( r = 0 \), because

\[ 99 = \overline{-33} \cdot \overline{3} + \overline{0} \]

\[ 0 \leq \frac{0}{r} < \frac{3}{b} \]

d. \( q = -33 \) and \( r = 0 \), because

\[ -99 = \overline{-33} \cdot \overline{3} + \overline{0} \]

\[ 0 \leq \frac{0}{r} < \frac{3}{b} \]

e. \( q = 0 \) and \( r = 0 \), because

\[ 0 = \overline{0} \cdot \overline{3} + \overline{0} \]

\[ 0 \leq \frac{0}{r} < \frac{3}{b} \]
Definition. Let $a, b \in \mathbb{Z}$ with $b > 0$

there is only one pair $(q, r)$ such that

- $a = qb + r$
- $0 \leq r < b$

We define the operations div and mod by

$$a \text{ div } b = q \quad \text{and} \quad a \mod b = r$$

35.2

a. Since $100 = \frac{33 \cdot 3}{q} + \frac{1}{r}$ we have

$$100 \text{ div } 3 = 33 \quad \text{and} \quad 100 \mod 3 = 1$$

b. Since $-100 = \frac{-34 \cdot 3}{q} + \frac{2}{r}$ we have

$$-100 \text{ div } 3 = -34 \quad \text{and} \quad -100 \mod 3 = 2$$

c. Since $99 = \frac{-33 \cdot 3}{q} + \frac{0}{r}$ we have

$$99 \text{ div } 3 = 33 \quad \text{and} \quad 99 \mod 3 = 0$$

d. Since $-99 = \frac{-33 \cdot 3}{q} + \frac{0}{r}$ we have

$$-99 \text{ div } 3 = -33 \quad \text{and} \quad -99 \mod 3 = 0$$

e. Since $0 = \frac{0 \cdot 3}{q} + \frac{0}{r}$ we have

$$0 \text{ div } 3 = 0 \quad \text{and} \quad 0 \mod 3 = 0$$

35.3

a. $N = 17, 18, 19, 20$, because

- $100 = 5 \cdot 17 + 15$ implies $100 \text{ div } 17 = 5$,
- $100 = 5 \cdot 18 + 10$ implies $100 \text{ div } 18 = 5$,
• $100 = 5 \cdot 19 + 5$ implies $100 \div 19 = 5$,
• $100 = 5 \cdot 20 + 0$ implies $100 \div 20 = 5$.

b. $N = 50, 51, 52, \ldots, 59$.

c. $N = 19, 95$

d. $N = 15, 25, 35, 45, \ldots$, because
• $15 = 1 \cdot 10 + 5$ implies $15 \mod 10 = 5$,
• $25 = 2 \cdot 10 + 5$ implies $25 \mod 10 = 5$,
• $35 = 3 \cdot 10 + 5$ implies $35 \mod 10 = 5$,
• $45 = 4 \cdot 10 + 5$ implies $45 \mod 10 = 5$.

: 

e. $N = -24, -23, -22, -21, -20$

35.4

a. Let $a, b$ be positive integers.
If $a \mod b = b \mod a$, then $a = b$.
We will show: "If $a \neq b$, then $a \mod b \neq b \mod a$.”
The proof of this statement is statement is logically equivalent to the proof of the statement
" If $a \mod b = b \mod a$, then $a = b$.”
Proof. If $a \neq b$, then $a < b$, or $a > b$.
Case $a < b$. Then $a = 0 \cdot b + \underbrace{a}_{\text{mod } b}$, therefore $a \mod b = a$ while $b = \overbrace{\tilde{q} \cdot a + \overbrace{\tilde{r} = a \mod b}^{b \text{ mod } a}}^{b \mod a}$ and $\tilde{r} < a$. Therefore $a \mod b \neq b \mod a$.
Case $a > b$. Then $b \mod a = b$ while $a \mod b < b$. Therefore $a \mod b \neq b \mod a$.

b. Let $a, b$ be positive integers.
If $a \div b = b \div a$, then $a = b$.
We will show: "If $a \neq b$, then $a \div b \neq b \div a$.”
The proof of this statement is statement is logically equivalent to the proof of the statement
" If $a \div b = b \div a$, then $a = b$.”
Proof. If \( a \neq b \), then \( a < b \), or \( a > b \).

**Case** \( a < b \). Then \( a = 0 \cdot b + \frac{a}{q} \), therefore \( a \div b = 0 \) while \( b = q \cdot a + r \) and \( r < a \). Because \( a < b \) we have \( q \neq 0 \) (otherwise \( b = 0 \cdot a + b \) would implie \( b < a \), however by the assumption \( a < b \)). Thus \( b \div a \neq 0 \) Therefore \( a \div b \neq b \div a \).

**Case** \( a > b \). Analogous

35.5

**a.** Let \( a, b \) be positive integers.

The statement ”If \( a < b \), then \( a \div c < b \div c \)” is not true:

**Counterexample.** \( a = 11, b = 12, \) and \( c = 5 \). Then \( a < b \), but \( a \div c = b \div c \).

**b.** Let \( a, b \) be positive integers.

The statement ”If \( a < b \), then \( a \mod c < b \mod c \)” is not true:

**Counterexample.** \( a = 9, b = 11, \) and \( c = 10 \). Then \( a < b \), but \( a \mod c \neq b \mod c \).