Joseph Fourier makes waves

NOTE: Most of the material here is standard, but some of it (mostly in the existence part), while I assume it is common knowledge, I have never seen done exactly as I do it here. Perhaps there is a reason.

1 Basics

We consider the wave equation in \( n \)-space variables:

\[
\frac{\partial^2}{\partial t^2} u = c^2 \Delta u.
\]

Here \( c > 0 \) is a constant. More specifically, we will consider the Cauchy problem

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} u &= c^2 \Delta u, \\
u(x, 0) &= f(x), \\
\frac{\partial}{\partial t} u(x, 0) &= g(x),
\end{align*}
\]

where \( f, g \) are given and will satisfy a number of possible assumptions.

We begin assuming that there exists \( u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \in C^2(\mathbb{R}^n \times \mathbb{R}) \) satisfying \( \frac{\partial^2}{\partial t^2} u - c^2 \Delta u = 0 \). We notice:

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} u_t^2 \right) = \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} (u_t u_{x_j}) - u_{tx_j} u_{x_j} \right)
\]

\[
= \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} (u_t u_{x_j}) - \frac{\partial}{\partial t} \left( \frac{1}{2} u_{x_j}^2 \right) \right)
\]

\[
= \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} (u_t u_{x_j}) \right) - \frac{\partial}{\partial t} \left( \frac{1}{2} \sum_{j=1}^{n} (u_{x_j}^2) \right).
\]

Thus,

\[
0 = u_t (\frac{\partial^2}{\partial t^2} u - c^2 \Delta u)
\]

\[
= -c^2 \sum_{j=1}^{n} \left( \frac{\partial}{\partial x_j} (u_t u_{x_j}) \right) + \frac{\partial}{\partial t} \left( \frac{1}{2} u_t^2 + \frac{c^2}{2} \sum_{j=1}^{n} (u_{x_j}^2) \right)
\]

For simplicity, let \( \nabla u \) be the gradient of \( u \) with respect to the space variables,

\[
\nabla u(x, t) = \left( \frac{\partial u}{\partial x_1}(x, t), \ldots, \frac{\partial u}{\partial x_n}(x, t) \right).
\]
Let us also introduce the vector valued \( W = (W_1, \ldots, W_{n+1}) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R} \) by

\[
W_j = -c^2 \left( \frac{\partial}{\partial x_j} (u_t u_x j), \quad j = 1, \ldots, n,
\right)
\]
\[
W_{n+1} = \frac{1}{2} u_t^2 + \frac{c^2}{2} \sum_{j=1}^{n} u_{x j}^2 = \frac{1}{2} \left( u_t^2 + c^2 |\nabla u|^2 \right).
\]

The vector \( W \) can be briefly written out as:

\[
W = \left( -c^2 u_t \nabla u, \frac{1}{2} \left( u_t^2 + c^2 |\nabla u|^2 \right) \right).
\]

With this notation, (2) can be written in the form

\[
\text{div}_{(x,t)} W(x, t) = 0.
\]

Notice \( W \in C^1(\mathbb{R}^n \times \mathbb{R}) \). Let \( R, T \) be real numbers, \( 0 \leq T \leq R/c \). Let

\[
G = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq t \leq T, |x| \leq R - ct \}.
\]

Then, by Gauss’ divergence theorem,

\[
0 = \int_G \text{div}_{(x,t)} W(x, t) \, dx \, dt = \int_{\partial G} W(x, t) \cdot n(x, t) \, dS,
\]

where \( n(x, t) \) denotes the outer normal to \( \partial G \) at \((x, t)\). The boundary of \( G \) can be decomposed into three parts; \( \partial G = A \cup B \cup C \), where

\[
A = \{(x, 0) : |x| \leq R \}, B = \{(x, t), 0 \leq t \leq T, |x| = R - c t \}, C = \{(x, T) : |x| \leq R - c T \}.
\]

On \( A \) the outer normal is given by \( n(x, 0) = (0, -1) \) for all \((x, 0) \in A\); on \( C \) it is \( n(x, T) = (0, 1) \) for all \((x, T) \in C\). The surface \( B \) can be described by

\[
\Phi(x, t) := |x|^2 - (R - ct)^2 = 0;
\]

thus a normal direction is given by

\[
\nabla \Phi(x, t) = (2x, 2c(R - ct)) = (2x, 2c|x|)
\]

since \(|x| = R - ct \) on \( B \). If \(|x| \) increases while keeping \( t \) fixed, the point \((x, t)\) will be outside of \( G \), moreover \( \Phi \) also increases. Thus \( \Phi \) increases as we move to the outside of \( G \) from the boundary, implying that the gradient of \( \Phi \) (which points in the direction of maximum increase) points outwards. So the outer normal on \( B \) is

\[
n(x, t) = \frac{\nabla \Phi(x, t)}{|\nabla \Phi(x, t)|} = \frac{1}{\sqrt{1 + c^2}} \left( \frac{x}{|x|}, c \right).
\]
Thus

\[ 0 = \int_{\partial G} W(x,t) \cdot n(x,t) \, dS = -\int_{|x|<R} W_{n+1}(x,0) \, dx + \frac{1}{\sqrt{1+c^2}} \int_B W(x,t) \cdot \left( \frac{x}{|x|},c \right) \, dS \]
\[ + \int_{|x|<R-cT} W_{n+1}(x,T) \, dx. \]

Now, for every \( t \in \mathbb{R}, \rho \geq 0, \)

\[ \int_{|x|\leq\rho} W_{n+1}(x,t) \, dx = \frac{1}{2} \int_{|x|\leq\rho} (|\nabla u(x,t)|^2 + u_t(x,t)^2) \, dx. \]

Concerning the integral over \( B, \) we have

\[ \int_B W(x,t) \cdot \left( \frac{x}{|x|},c \right) \, dS = \int_B \left( -c^2 u_t \nabla u \cdot \frac{x}{|x|} + \frac{c}{2} \left( u_t^2 + c^2 |\nabla u|^2 \right) \right) \, dS. \]

The integrand satisfies

\[ -c^2 u_t \nabla u \cdot \frac{x}{|x|} + \frac{c}{2} \left( u_t^2 + c^2 |\nabla u|^2 \right) = \frac{c}{2} \left( u_t^2 + c^2 |\nabla u|^2 - 2cu_t \nabla u \cdot \frac{x}{|x|} \right) \]
\[ \geq \frac{c}{2} \left( u_t^2 + c^2 |\nabla u|^2 - 2cu_t |\nabla u| \frac{x}{|x|} \right) \]
\[ = \frac{c}{2} \left( u_t^2 + c^2 |\nabla u|^2 - 2cu_t |\nabla u| \right) = \frac{c}{2} (c-u_t-c|\nabla u|)^2 \geq 0. \]

If we put it all together we get

\[ 0 = -\frac{1}{2} \int_{|x|\leq R} (|\nabla u(x,0)|^2 + u_t(x,0)^2) \, dx + \frac{1}{2} \int_{|x|\leq R-cT} (|\nabla u(x,T)|^2 + u_t(x,T)^2) \, dx + P, \]

where

\[ P = \frac{1}{\sqrt{1+c^2}} \int_B \left( -c^2 u_t \nabla u \cdot \frac{x}{|x|} + \frac{c}{2} \left( u_t^2 + c^2 |\nabla u|^2 \right) \right) \, dS \geq 0. \]

Thus

\[ \frac{1}{2} \int_{|x|\leq R} (|\nabla u(x,0)|^2 + u_t(x,0)^2) \, dx = \frac{1}{2} \int_{|x|\leq R-cT} (|\nabla u(x,T)|^2 + u_t(x,T)^2) \, dx + P \]
\[ \geq \frac{1}{2} \int_{|x|\leq R-cT} (|\nabla u(x,T)|^2 + u_t(x,T)^2) \, dx. \]

We proved:

**Lemma 1** Let \( u \in C^2(\mathbb{R}^n \times \mathbb{R}) \) and satisfy \( u_{tt} = \Delta u. \) Then, for every \( R,T \in \mathbb{R}, \) \( 0 \leq T \leq R/c, \) we have

\[ \int_{|x|\leq R-cT} (|\nabla u(x,T)|^2 + u_t(x,T)^2) \, dx \leq \int_{|x|\leq R} (|\nabla u(x,0)|^2 + u_t(x,0)^2) \, dx. \]
Everything is translation invariant. Moreover, things are symmetric in \( t \) so what we did for \( 0 \leq t \leq T \) is also valid for \( 0 \geq t \geq -T \) (mutatis mutandis). Also, we can write again \( t \) for \( T \). Thus we can generalize immediately Lemma 1 to 

**Lemma 2** Let \( u \in C^2(\mathbb{R}^n \times \mathbb{R}) \) and satisfy \( u_{tt} = \Delta u \). Then, for every \( x_0 \in \mathbb{R}^n \), \( R, t \in \mathbb{R}, 0 \leq |t| \leq R/c \), we have

\[
\int_{|x-x_0| \leq R-c|t|} (|\nabla u(x,t)|^2 + u_t(x,t)^2) \, dx \leq \int_{|x-x_0| \leq R} (|\nabla u(x,0)|^2 + u_t(x,0)^2) \, dx.
\]

Here are some immediate consequences of this result. We’ll state them in the form of propositions, but they are quite immediate. It may be convenient to notice that if \( u(x,0) = f(x), u_t(x,0) = g(x) \), then the inequality of Lemma 2 can be written in the form

\[
(3) \int_{|x-x_0| \leq R-c|t|} (|\nabla u(x,t)|^2 + u_t(x,t)^2) \, dx \leq \int_{|x-x_0| \leq R} (|\nabla f(x)|^2 + g(x)^2) \, dx,
\]

valid for \( |t| \leq R/c \).

**Proposition 3** Let \( u, v \in C^2(\mathbb{R}^n \times \mathbb{R}) \) satisfy the wave equation; i.e., \( u_{tt} - \Delta u = 0 = v_{tt} - \Delta v \) and assume that \( u(x,0) = v(x,0) \) and \( u_t(x,0) = v_t(x,0) \) for \( |x-x_0| \leq R \), for some \( x_0 \in \mathbb{R}^n \), \( R > 0 \). Then \( u(x,t) = v(x,t) \) for \( |x-x_0| \leq R-c|t|, |t| \leq R/c \).

**Proof.** Replacing \( u \) by \( u-v \), we can assume \( v = 0 \), prove that \( u(x,0) = u_t(x,0) = 0 \), in \( |x-x_0| \leq R \) implies \( u(x,t) = 0 \) in \( |x-x_0| + c|t| \leq R \). Let \( |t| \leq R/c \). By Lemma 2

\[
\int_{|x-x_0| \leq R-c|t|} (|\nabla u(x,t)|^2 + u_t(x,t)^2) \, dx = 0;
\]

hence (since non-negative)

\[
\int_{|x-x_0| \leq R-c|t|} (|\nabla u(x,t)|^2 + u_t(x,t)^2) \, dx = 0
\]

implying that \( u_t(x,t) = 0, \nabla u(x,t) = 0 \) in \( |x-x_0| < R-c|t| \). The “double cone” \( \{ (x,t) \in \mathbb{R}^n \times \mathbb{R} : |x-x_0| + c|t| < R \} \) being connected (in fact, convex), it follows that \( u \) is constant in that region, hence 0 (being 0 for \( |x-x_0| < R \)). By continuity, all strict inequalities can be replaced by non-strict ones. ■

**Proposition 4** (Uniqueness) Let \( f \in C^2(\mathbb{R}^n), g \in C^1(\mathbb{R}) \). The Cauchy problem (1) has at most one solution \( u \in C^2(\mathbb{R}^n \times \mathbb{R}) \).

**Proof.** If \( u, v \) both satisfy the same Cauchy problem, then we can apply Proposition 3; the hypotheses hold for every \( R > 0 \). The proposition follows. ■

Incidentally, if \( u \) solves the Cauchy problem and is in \( C^2 \), then \( a \) fortiori \( f \in C^2 \) and \( g \in C^1 \).
Proposition 5 Assume \( u \in C^2(\mathbb{R}^n \times \mathbb{R}) \) solves the Cauchy problem (1) with 
\( f, g \) of compact support. Then \( u(t) : x \mapsto u(x,t) \) has compact support for every 
\( t \in \mathbb{R} \). Specifically, if \( \text{supp} \ f \cup \text{supp} \ g \subset \{|x| \leq \rho|t|\} \) (for some \( \rho \geq 0 \)), then \( \text{supp} \ u(t) \subset \{|x| \leq \rho + c|t|\} \).

Proof. Exercise.

Proposition 6 Assume \( u \in C^2(\mathbb{R}^n \times \mathbb{R}) \) solves the Cauchy problem (1) with 
\( f, g \) satisfying 
\[
\int_{\mathbb{R}^n} (|\nabla f(x)|^2 + |g(x)|^2) \, dx < \infty.
\]
Then 
\[
E(u)(t) := \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla u(x,t)|^2 + u_t(x,t)^2) \, dx = \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla f(x)|^2 + |g(x)|^2) \, dx < \infty
\]
for all \( t \in \mathbb{R} \).

Proof. Exercise.

Remark. The quantity 
\[
E(u)(t) := \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla u(t)|^2 + u_t(t)^2) \, dx
\]
is referred to as the energy of the solution of the wave equation, and Proposition 6 is known as conservation of energy.

2 The Fourier transform jumps into action

2.1 The straightforward part

We will try to figure out the inverse Fourier transform of the function \( \xi \mapsto \sin (|\xi|/|\xi|) \). As we shall see, it will suffice to know it for odd dimensions. Let us begin noticing that if \( u, v \) are solutions of the wave equation satisfying the initial conditions 
\[
u(0) = 0, \quad u_t(0) = g; \quad v(0) = 0, \quad v_t(0) = f,
\]
then \( w = u + v_t \) is a solution of the wave equation satisfying \( w(0) = f, \, w_t(0) = g \).

In fact, that \( w \) solves the wave equation is clear. Moreover, 
\[
w(0) = u(0) + v_t(0) = f, \quad w_t(0) = u_t(0) + v_{tt}(0) = g + \Delta v(0) = g + 0 = g.
\]

Notice We are following the convention that if \( F : U \times I \to S \), where \( U \subset \mathbb{R}^n \), 
\( I \subset \mathbb{R} \) and \( S \) is a set, then for each \( t \in I \), \( F(t) : U \to S \) defined by \( F(t)(x) = F(x,t) \).
Returning to our calculations, what they show is that it suffices to solve the Cauchy problem (1) with \( f = 0 \). We will work formally for a while, get a formula for the solution, then verify it actually is a solution. For this we introduce \( \hat{u} \), the Fourier transform of \( u \) with respect to the \( x \)-variable. We assume it exists. We can make this assumption (which has to do with \( u \) not growing too fast as \(|x| \to \infty|\) for two reasons. Reason 1, if we cut off \( u \) for \( t = 0 \), making it zero for \(|x| > R\), that only affects the solution at level \( t \) for \(|x| > R - c|t|\). Taking \( R \) large enough, we see that at each level \( t \), the solution at \((x, t)\) can be obtained from compactly supported initial values. Reason 2, we are working formally.

To repeat, we let
\[
\hat{u}(\xi, t) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x, t) \, dx.
\]
Then one sees that
\[\hat{u}_{tt} + c^2|\xi|^2 \hat{u} = 0;\]
this is a second order, ordinary differential equation for \( \hat{u} \) as a function of \( t \) (for each fixed \( \xi \in \mathbb{R}^n \)) with the general solution
\[\hat{u}(\xi, t) = A(\xi) \cos c|\xi|t + B(\xi) \sin c|\xi|t.\]
The initial conditions work out to
\[\hat{u}(\xi, 0) = 0, \quad \hat{u}_t(\xi, 0) = \hat{g}(\xi),\]
from which we get \( A(\xi) = 0, \quad B(\xi) = \hat{g}(\xi)/(c|\xi|) \). That is,
\[\hat{u}(\xi, t) = \frac{\sin c|\xi|t}{c|\xi|} \hat{g}(\xi).\]
Now suppose we know a function \( s \) such that \( \hat{s}(\xi) = \sin |\xi|/|\xi| \). (As it turns out, if \( n > 1 \) then \( s \) is only a distribution, but let’s not worry about that yet.) A simple change of variables shows that if we define
\[s_t(x) = t(tc)^{-n} s \left( \frac{x}{ct} \right),\]
then \( \hat{s}_t(\xi) = \sin(c|\xi|t)/c|\xi| \), hence
\[\hat{u}(\xi, t) = \hat{s}_t(\xi) \hat{g}(\xi).\]
Inverting the Fourier transform,
\[u(x, t) = (2\pi)^{-\frac{n}{2}} \hat{s}_t * g(x) = \frac{t}{(\sqrt{2\pi ct})^n} \int_{\mathbb{R}^n} s \left( \frac{x - y}{ct} \right) g(y) \, dy.\]
Well, things won’t work out exactly this way because \( s \) will turn out to be in \( \mathcal{S}' \), not a function. Except if \( n = 1 \).
**Exercise.** Assume \( n = 1 \) and define \( s(x) = \sqrt{\pi/2} \) if \(|x| < 1\), \( s(x) = 0 \) otherwise. Show that \( \hat{s}(\xi) = \sin \xi/\xi = \sin |\xi|/|\xi| \) and see that, in this case, equation (5) becomes

\[
u(x,t) = \frac{1}{2c} \int_{y-c|t|}^{y+c|t|} g(y) \, dy.
\]

For the tedious fun of it, you might also want to check that if \( g \in C^1(\mathbb{R}) \) and you define a function \( \nu : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) this way, then \( \nu \in C^2(\mathbb{R} \times \mathbb{R}) \) and \( \nu_{tt} = c^2 \nu_{xx} \). There is a small difficulty due to the appearance of \(|t|\), which is not differentiable at 0.

So let us now try to find the inverse Fourier transform of \( \xi \mapsto \sin t|\xi|/|\xi| \), for \( n \) odd, \( n \geq 3 \). But before getting to it, I’ll develop some additional material about Fourier transforms, integrals, whatnot, so as to simplify a bit the computations.

### 2.2 Some results from the Fourier transform notes recalled. And maybe more

One thing we need to recall is the Fourier transform of the function \( x \mapsto e^{-a|x|^2} \), where \( a > 0 \). That is,

\[
(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{-a|x|^2} \, dx = (2a)^{-\frac{n}{2}} e^{-\frac{|\xi|^2}{4a}}.
\]

We’ll only need it for \( n = 1 \), in the form

(6)

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\rho \xi} e^{-\epsilon \rho^2} \, d\rho = \frac{1}{\sqrt{2\epsilon}} e^{-\xi^2/4\epsilon}.
\]

We also need a kernel result. For proofs I’ll refer to my notes on Fourier transforms. Here I’ll just state the result for \( n = 1 \). Let \( k_\epsilon : \mathbb{R} \rightarrow \mathbb{R} \) for each \( \epsilon > 0 \). Our model will be

\[
k_\epsilon(x) = (2\pi \epsilon)^{-\frac{1}{2}} e^{-\frac{x^2}{4\pi \epsilon}}
\]

I am leaving it as an exercise to verify that this concrete \( k_\epsilon \) satisfies all the properties I am about to list. Assume \( k_\epsilon \) is a function from \( \mathbb{R} \) to \( \mathbb{R} \) for all \( \epsilon > 0 \) satisfying:

1. \( k_\epsilon(x) \geq 0 \) for all \( x \in \mathbb{R} \).
2. \( \int_{-\infty}^{\infty} k_\epsilon(x) \, dx = 1 \).
3. For each \( \delta > 0 \),

\[
\lim_{\epsilon \to 0^+} \int_{|x| > \delta} k_\epsilon(x) \, dx = 0.
\]

Under these circumstances the following is true.
Lemma 7 Let $f : \mathbb{R} \to \mathbb{C}$ be measurable and bounded. Assume $f$ is continuous at $x_0 \in \mathbb{R}$. Then
$$\lim_{\epsilon \to 0^+} k_\epsilon \ast f(x_0) = f(x_0).$$

In the notes on Fourier transforms I assume $f$ is continuous everywhere, but the same proof applies. It is easy enough that it can be quickly recalled here. Since $f$ is bounded, there is $M$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Since $f$ is continuous at $x_0$, given $\eta > 0$ there is $\delta > 0$ such that $|f(x) - f(x_0)| < \eta/2$ if $|x - x_0| < \delta$. Since $\int_{\mathbb{R}} k_\epsilon(x) \, dx = 1$ we can write
$$|k_\epsilon \ast f(x_0) - f(x_0)| = \left| \int_{-\infty}^{\infty} k_\epsilon(x)f(x_0 - x) \, dx - f(x_0) \right| = \left| \int_{-\infty}^{\infty} k_\epsilon(x)[f(x_0 - x) - f(x_0)] \, dx \right|
\leq \int_{-\infty}^{\infty} \left| k_\epsilon(x)f(x_0 - x) - f(x_0) \right| \, dx
= \int_{|x| < \delta} k_\epsilon(x)|f(x_0 - x) - f(x_0)| \, dx + \int_{|x| > \delta} k_\epsilon(x)|f(x_0 - x) - f(x_0)| \, dx
< \frac{\eta}{2} \int_{|x| < \delta} k_\epsilon(x) \, dx + 2M \int_{|x| > \delta} k_\epsilon(x) \, dx
\leq \frac{\eta}{2} \int_{-\infty}^{\infty} k_\epsilon(x) \, dx + 2M \int_{|x| > \delta} k_\epsilon(x) \, dx
= \frac{\eta}{2} + 2M \int_{|x| > \delta} k_\epsilon(x) \, dx.
$$
Since
$$\lim_{\epsilon \to 0^+} \int_{|x| > \delta} k_\epsilon(x) \, dx = 0,$$
there is $\epsilon_0 > 0$ such that $2M \int_{|x| > \delta} k_\epsilon(x) \, dx < \eta/2$ if $0 < \epsilon < \epsilon_0$. The result follows.

2.3 Bochner’s formula; the Fourier transform of a radial function

This section is devoted to proving a classical formula for the Fourier transform of a radial function. A Bessel function appears in it, so let us recall (learn?) that if $\nu \geq 0$, $J_\nu$ is the universal symbol for the Bessel function of the first kind of order $\nu$; that is, it is the one and only solution of the equation
$$z^2 J''(z) + z J'(z) + (z^2 - \nu^2) J(z) = 0$$
that is bounded at 0 and is normalized by
$$\lim_{z \to 0^+} z^{-\nu} J_\nu(z) = \frac{1}{2\pi \Gamma(\nu + 1)}.$$  

For $\nu = k + \frac{1}{2}$ (which will be our case), there are explicit formulas for these functions in terms of trigonometric functions. For example, and we’ll need this later on
$$J_\frac{1}{2}(z) = \sqrt{\frac{\pi}{2}} \frac{\sin z}{\sqrt{z}}.$$
We will also need the following recurrence formula for Bessel functions:

\[ \frac{d}{dz} \left( z^{-\nu} J_{\nu}(z) \right) = -z^{-\nu} J_{\nu+1}(z). \]

One of the most comprehensive, and perhaps still the best, reference for Bessel functions is G.N. Watson’s monumental *Theory of Bessel Functions*.

A radial function is, of course, one that only depends on \(|x|\). That is, \( f : \mathbb{R}^n \to \mathbb{C} \) is radial iff there exists \( f_0 : [0, \infty) \to \mathbb{C} \) such that \( f(x) = f_0(|x|) \). To avoid too much notation I will abuse language (or notation) a bit and simply write \( f(x) = f(|x|) \) if \( f \) is radial, or even \( f(x) = f(r) \) where \( r = |x| \).

**Theorem 8 (Bochner’s Formula)** Let \( f \in L^1(\mathbb{R}^n) \) be radial. Then

\[ \hat{f}(\xi) = |\xi|^{-\frac{n-2}{2}} \int_0^\infty \rho^{\frac{n-1}{2}} J_{n-2}(\rho|\xi|) f(\rho) \, d\rho. \]

**Proof.** Using

\[ \int_{\mathbb{R}^n} g(x) \, dx = \int_0^\infty \rho^{n-1} \int_{S^{n-1}} g(\rho \omega) \, dS_\omega \]

with \( g(x) = f(x)e^{-ix \cdot \xi} \), thus \( g(\rho \omega) = f(\rho)e^{-i\rho \xi \cdot \omega} \), we can write the Fourier transform of \( f \) in the form

\[ \hat{f}(\xi) = (2\pi)^{-\frac{\frac{n-2}{2}}{2}} \int_0^\infty \rho^{\frac{n-1}{2}} f(\rho) \int_{S^{n-1}} e^{-i\rho \xi \cdot \omega} \, dS_\omega \, d\rho. \]

We will prove below that for \( x \in \mathbb{R}^n \),

\[ \int_{S^{n-1}} e^{ix \cdot \omega} \, dS_\omega = (2\pi)^{-\frac{\frac{n-2}{2}}{2}} |x|^{-\frac{n-2}{2}} J_{n-2}(|x|). \]

Using this in (11) with \( x = -\rho \xi \), (10) follows at once. The Theorem is proved, modulo the proof of (12).

**Proof of (12).** Set

\[ I(x) = \int_{S^{n-1}} e^{ix \cdot \omega} \, dS_\omega. \]

**Lemma 9** The function \( I \) is radial; in particular if \( T \) is an \( n \times n \) orthogonal matrix, \( I(Tx) = I(x) \).

**Proof.** It is not hard to see that being radial (i.e., being a function of \(|x|\)) is equivalent to satisfying \( I(Tx) = I(x) \) for all orthogonal matrices \( T \). Now

\[ I(Tx) = \int_{S^{n-1}} e^{iT \cdot x \cdot \omega} \, dS_\omega = \int_{S^{n-1}} e^{ix \cdot T^\dagger \omega} \, dS_\omega. \]
The surface measure on the unit sphere \( S^{n-1} \) is rotation invariant, so changing variables by \( \omega' = T_t \omega \) one gets that the last displayed integral equals \( \int_{S^{n-1}} e^{ix \cdot \omega} dS'_\omega = I(x) \).

Now

\[
\Delta I(x) = \int_{S^{n-1}} \Delta_t e^{ix \cdot \omega} dS_\omega = \int_{S^{n-1}} (-i|\omega|)^2 e^{ix \cdot \omega} dS_\omega = -I(x),
\]

since \(|\omega|^2 = 1\) for \( \omega \in S^{n-1} \). In other words, \( \Delta I + I = 0 \). But, as seen, \( I \) is radial; writing with some abuse of notation \( I(x) = I(r) \) where \( r = |x| \), and recalling the form of the Laplacian of a radial function

\[
\Delta f(x) = \frac{d^2f}{dr^2} + \frac{n-1}{r} \frac{df}{dr},
\]

\( I \) satisfies

\[
I''(r) + \frac{n-1}{r} I' + I = 0.
\]

Define \( J(r) = r^{n-2} I(r) \), so \( I(r) = r^{n-2} J(r) \), and we see that

\[
r^2 J''(r) + r J'(r) + \left( r^2 - \left( \frac{n-2}{2} \right)^2 \right) J(r) = 0.
\]

In other words, \( J \) satisfies a Bessel equation of order \( \nu = (n-2)/2 \). Because \( I \) is clearly \( C^\infty \) in \( x \), \( J \) is a solution that is nice at 0, so it is a constant times \( J_\nu \). That is

\[
I(x) = c_n |x|^{-\nu} J_\nu(|x|),
\]

for some constant \( c_n \). Going to Watson’s monumental *Theory of Bessel Functions*, we learn that \( \lim_{r \to 0^+} r^{-\nu} J_\nu(r) \) is taken to be \( 1/(2^{\nu} \Gamma(\nu + 1)) \). On the other hand

\[
I(0) = \int_{S^{n-1}} dS_\omega = \omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.
\]

Thus

\[
c_n = \frac{2\pi^{\frac{n}{2}} 2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = (2\pi)^{\frac{n}{2}}.
\]

We proved

\[
\int_{S^{n-1}} e^{ix \cdot \omega} dS_\omega = (2\pi)^{\frac{n}{2}} |x|^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(|x|),
\]

which is (12).

On consequence of Bochner’s formula (which can also be proved directly without major trouble) is that the Fourier transform of a radial function is radial. Radial functions are even, for an even function there is no difference between applying the Fourier transform or the inverse Fourier transform. We
will really need Bochner’s formula for the inverse Fourier transform, specifically
the following formula: let $h \in L^1(\mathbb{R}^n)$ be radial. Then

$$F^{-1}h(x) = |x|^{-\frac{n+2}{2}} \int_0^\infty \rho^{\frac{n}{2}} J_{\frac{n+2}{2}}(\rho |x|) h(\rho) \, d\rho.$$  \hspace{1cm} (14)

2.4 Inverting $\xi \mapsto \sin t|\xi|/|\xi|$

We will assume $t > 0$. We also assume, until further notice, that $n$ is odd, $n \geq 3$. Because $\xi \mapsto \sin t|\xi|/|\xi|$ clearly is in $S'(\mathbb{R}^n)$ for each $t > 0$ (also for $t \leq 0$; it is a bounded function of $\xi$), there is $S_t \in S'(\mathbb{R}^n)$ such that $\hat{S_t}(\xi) = \sin t|\xi|/|\xi|$. We have, if $\varphi \in S(\mathbb{R}^n),$

$$\langle S_t, \varphi \rangle = \left\langle \hat{S_t}, F^{-1}(\varphi) \right\rangle = \int_{\mathbb{R}^n} \frac{\sin t|\xi|}{|\xi|} F^{-1}(\varphi)(\xi) \, d\xi$$

$$= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \frac{\sin t|\xi|}{|\xi|} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \varphi(x) \, dx \, d\xi.$$

We would like to change the order of integration, but the $\xi$ integral would not converge. A standard trick is to introduce some convergence factor such as $e^{-|x|^2}$ and let $\epsilon \to 0^+$, or replace the $\xi$ integral by the limit as $R \to \infty$ of the integral over $|\xi| \leq R$. Then one can change the order of integration, hoping that at the end the limit (for $\epsilon \to 0^+$ or $R \to \infty$) will be “easy” to calculate. I prefer the convergence factor approach, so we continue our evaluation by

$$\langle S_t, \varphi \rangle = (2\pi)^{-\frac{n}{2}} \lim_{\epsilon \to 0^+} \int_{|\xi| \leq R} e^{-\epsilon|\xi|^2} \frac{\sin t|\xi|}{|\xi|} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \varphi(x) \, dx \, d\xi$$

$$= (2\pi)^{-\frac{n}{2}} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} \varphi(x) \left( \int_{\mathbb{R}^n} e^{-\epsilon|\xi|^2} \frac{\sin t|\xi|}{|\xi|} \, d\xi \right) \, dx.$$

Now $(2\pi)^{-n/2}$ times the inner integral, namely,

$$(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\epsilon|\xi|^2} \frac{\sin t|\xi|}{|\xi|} \, d\xi$$

is the inverse Fourier transform of the radial function $\xi \mapsto e^{-\epsilon|\xi|^2} \frac{\sin t|\xi|}{|\xi|}$ thus, by Bochner’s formula (14), we have

$$\langle S_t, \varphi \rangle = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} \varphi(x) \int_0^\infty e^{-\epsilon\rho^2} \rho^\frac{n-2}{2} \sin t \rho |x|^{-\frac{n-2}{2}} J_{\frac{n+2}{2}}(\rho |x|) \, d\rho \, dx.$$

For $r > 0$, $\nu > 0$ set

$$I_{\nu, t}(r) = r^{-\nu} \int_0^\infty e^{-\epsilon\rho^2} \rho^\nu \sin t \rho I_{\nu}(\rho r) \, d\rho.$$  \hspace{1cm} (15)

$I_{\nu}$ also depends on $t$, but I’ll keep $t > 0$ fixed for a while, so it doesn’t have to be explicitly mentioned in the notation. Then

$$\langle S_t, \varphi \rangle = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n} \varphi(x) I_{\frac{n-2}{2}, t}(|x|) \, dx.$$  \hspace{1cm} (16)
We need the following properties of the function $I_{\nu, \epsilon}$, conveniently packaged in
the form of lemmas.

**Lemma 10** Assume $\nu > 0$. The function $I_{\nu, \epsilon} : (0, \infty) \to \mathbb{R}$ is differentiable
and bounded. It satisfies the recurrence relation

$$I_{\nu}(r) = -\frac{1}{r} \frac{dI_{\nu-1}}{dr}(r)$$

if $\nu - 1 > 0$.

**Proof.** Differentiability is fairly obvious; the only question is boundedness. We
can write

$$I_{\nu, \epsilon}(r) = \int_0^\infty e^{-\epsilon \rho^2} \rho^{2\nu} \sin t \rho \, d\rho$$

and, as we can learn from Dr. Watson's masterwork, the function $z \mapsto z^{-\nu} J_\nu(z)$
is bounded for $z > 0$ (and goes to 0 of order $z^{-\nu-\frac{1}{2}}$ as $z \to \infty$). The integral
is thus bounded because we are integrating a bounded function times the inte-
grable function $\rho \mapsto e^{-\epsilon \rho^2}$.

By the recurrence relation (9),

$$\frac{\partial}{\partial r} (r^{-\nu} J_\nu(\rho r)) = \rho^\nu \frac{\partial}{\partial r} ((\rho r)^{-\nu} J_\nu(\rho r)) = \rho^\nu \frac{\partial}{\partial (\rho r)} ((\rho r)^{-\nu} J_\nu(\rho r)) \frac{\partial (\rho r)}{\partial r}$$

Thus

$$\frac{1}{r} \frac{dI_{\nu, \epsilon}}{dr}(r) = \frac{1}{r} \int_0^\infty e^{-\epsilon \rho^2} \rho^\nu \sin t \rho \frac{\partial}{\partial \rho} (r^{-\nu} J_\nu(\rho r)) \, d\rho$$

$$= -r^{-\nu-1} \int_0^\infty e^{-\epsilon \rho^2} \rho^{\nu+1} \sin t \rho J_{\nu+1}(\rho r) \, d\rho = -I_{\nu+1, \epsilon}(r).$$

Replacing $\nu$ by $\nu - 1$ we obtain (17). \[\square\]

For the next lemma, I'll define $\mathcal{S}(0, \infty)$ to consist of all functions in $C^1(0, \infty)$
such that $\lim_{r \to 0^+} \psi^{(k)}(r)$ exists for all $k \in \mathbb{N} \cup \{0\}$ and such that
$\sup_{r \in [0, \infty)} r^m |\psi^{(k)}(r)| < \infty$ for all $m, k \in \mathbb{N} \cup \{0\}$. We say that $\psi : [0, \infty)$ has
a zero of order $m \in \mathbb{N}$ at 0 iff $|\psi(r)| \leq Cr^m$ near 0 for some constant $C$. If $\psi \in \mathcal{S}(0, \infty)$ has a zero of order $m > 1$ at 0, then $\psi'$ has a zero of order $m - 1$ at
0. Moreover, $r \mapsto \psi(r)/r \in \mathcal{S}(0, \infty)$ and, if $m > 1$, has a zero of order $m - 1$ at
0. I may add proofs of these facts as an appendix; they are simple introductory
analysis (or modern analysis) level exercises.

$^1J_\nu$ is also defined and equal to $J_{-\nu}$ if $\nu < 0$, but I don’t want to get involved with negative orders.
Lemma 11 Let \( \nu = k + \frac{1}{2} \), where \( k \in \mathbb{N} \cup \{0\} \). Let \( \psi \in \mathcal{S}[0, \infty) \) have a zero of order \( 2k + 1 \) at 0. For \( \ell = 0, \ldots, k \), the function \( \left( \frac{1}{r} \frac{d}{dr} \right)^\ell \psi(r) \in \mathcal{S}[0, \infty) \) and has a zero of order \( 2(k - \ell) + 1 \) at 0. Moreover

\[
\int_0^\infty r I_{\nu, \ell}(r) \psi(r) \, dr = \int_0^\infty r I_{2, \ell}(r) \left( \frac{1}{r} \frac{d}{dr} \right)^k \psi(r) \, dr. \tag{18}
\]

Proof. By the facts mentioned before this lemma, \( \psi' \in \mathcal{S}[0, \infty) \) and has a zero of order \( 2k \) at 0; thus \( \psi'/r \in \mathcal{S}[0, \infty) \) and has a zero of order \( 2k - 1 = 2(k - 1) + 1 \) at 0. By induction, \( \left( \frac{1}{r} \frac{d}{dr} \right)^k \psi(r) \in \mathcal{S}[0, \infty) \) has a zero of order \( 2(k - \ell) + 1 \) at 0 for \( \ell = 0, 1, \ldots, k \).

It remains to prove (18). If \( k = 0 \), there is nothing to prove, so assume \( k \geq 1 \), hence \( \nu > 1 \). By the recurrence relation (17),

\[
\int_0^\infty r I_{\nu, \ell}(r) \psi(r) \, dr = - \int_0^\infty I'_{\nu-1, \ell}(r) \psi(r) \, dr = - I_{\nu-1, \ell}(r) \psi(r) \bigg|_0^\infty + \int_0^\infty I_{\nu-1, \ell}(r) \psi'(r) \, dr.
\]

Because \( \psi \) is zero at 0 and goes fast to zero at infinity, the first term on the right is 0 and

\[
\int_0^\infty r I_{\nu, \ell}(r) \psi(r) \, dr = \int_0^\infty I_{\nu-1, \ell}(r) \psi'(r) \, dr \int_0^\infty r I_{\nu-1, \ell}(r) \left( \frac{1}{r} \psi'(r) \right) \, dr.
\]

Now \( \nu - 1 = k - \frac{1}{2} = (k - 1) + \frac{1}{2} \), \( \psi'/r \) has a zero of order \( 2(k - 1) + 1 \) at 0 so, if \( k - 1 \geq 1 \), we can repeat the process. After \( k \) iterations we obtain (18) \( \square \)

The next lemma contains the last semi-lengthy computations needed to invert the Fourier transform of \( \sin t|\xi|/|\xi| \). It is time to bring \( t \) back into the picture.

Lemma 12 Let \( h \in C([0, \infty)) \) and assume \( h \) is bounded. Then

\[
\lim_{\epsilon \to 0^+} \int_0^\infty r I_{\nu-1, \ell}(r) h(r) \, dr = \sqrt{\frac{\pi}{2}} h(t). \tag{19}
\]

Proof. Here is were we need to recall some calculations done in the Fourier Transform notes. We need the formula

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} e^{-\xi^2/4\epsilon} \, d\xi = \frac{1}{\sqrt{2\epsilon}} e^{-x^2/4\epsilon},
\]

which we mentioned above in (6). Recall also that \( J_{1/2}(z) = \sqrt{\frac{2}{\pi}} \frac{\sin z}{\sqrt{z}} \). With this we can calculate \( I_{1/2} \):

\[
I_{1/2}(r) = r^{-1/2} \int_0^{\infty} e^{-\epsilon \rho^2} \rho^{1/2} \sin t \rho J_{1/2}(\rho r) \, d\rho = \sqrt{\frac{2}{\pi}} r^{-1/2} \int_0^{\infty} e^{-\epsilon \rho^2} \rho^{1/2} \sin t \rho \frac{\sin \rho r}{\sqrt{\rho r}} \, d\rho
\]

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\[= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\rho^2} \sin \rho \sin \rho \, d\rho = \frac{1}{r\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\rho^2} \sin \rho \, d\rho \]
\[= \frac{1}{2r\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\rho^2} (\cos \rho (t-r) - \cos \rho (t+r)) \, d\rho \]
\[= \frac{1}{2r\sqrt{2\pi}} \Re \left( \int_{-\infty}^\infty e^{-\rho^2} \left( e^{i\rho (t-r)} - e^{i\rho (t+r)} \right) \, d\rho \right).\]

By (6), with \(\xi = t-r\) and with \(\xi = t+r\), we get
\[(20) \quad I_{1/2}(r) = \frac{1}{2r\sqrt{2\epsilon}} \left( e^{-\frac{(t+\epsilon r)^2}{\epsilon}} - e^{-\frac{(t+\epsilon r)^2}{\epsilon}} \right).\]

We are assuming \(t > 0\); due to this \(\lim_{\epsilon \to 0} e^{-\frac{(t+\epsilon r)^2}{\epsilon}} = 0\); the convergence being rather violent. By dominated convergence (or even Riemann integral results), the integral of it times \(rh(r)\) also goes to 0. We are left with
\[
\lim_{\epsilon \to 0+} \int_0^\infty r I_{a-1, \epsilon}(r) h(r) \, dr = \lim_{\epsilon \to 0+} \frac{1}{2\sqrt{2\epsilon}} \int_0^\infty e^{-\frac{(t+\epsilon r)^2}{\epsilon}} h(r) \, dr.
\]

The function \(h\) is assumed to be continuous and bounded on \([0, \infty)\), therefore it is continuous at \(t > 0\). It still will be continuous at \(t\) if we extend it as 0 to the negative axis so that we can apply Lemma 7 to conclude that
\[
\lim_{\epsilon \to 0+} \frac{1}{2\sqrt{2\epsilon}} \int_0^\infty e^{-\frac{(t-r)^2}{4\epsilon}} h(r) \, dr = \sqrt{\frac{\pi}{2}} \lim_{\epsilon \to 0+} \frac{1}{\sqrt{4\pi \epsilon}} \int_0^\infty e^{-\frac{(t-r)^2}{4\epsilon}} h(r) \, dr = \sqrt{\frac{\pi}{2}} h(t).
\]

This has been a long journey, but the end is now in sight. With \(\varphi \in S(\mathbb{R}^n)\) we will take
\[(21) \quad \psi(r) = r^{n-2} \int_{S^{n-1}} \varphi(r \omega) \, dS_\omega.
\]

We will also take now \(\nu = \frac{n-2}{2} = k + \frac{1}{2}\), where \(k = (n-3)/2\). The function \(\psi\) decreases very fast, together with its derivatives, at \(\infty\), because \(\varphi \in S(\mathbb{R}^n)\). It has obviously a zero of order \(n-2 = 2\nu\) at 0. Then
\[
\int_{\mathbb{R}^n} \varphi(x) I_{a-2}(|x|) \, dx = \int_0^\infty r^{n-1} I_{a-2}(r) \int_{S^{n-1}} \varphi(r \omega) \, dS_\omega = \int_0^\infty r I_a(r) \psi(r) \, dr;
\]

By (16), (18) in Lemma 11 and Lemma 12,
\[(S_t, \varphi) = \lim_{\epsilon \to 0+} \int_0^\infty r I_a(r) \left( \frac{1}{r} \frac{d}{dr} \right)^k \psi(r) \, dr = \sqrt{\frac{\pi}{2}} \left( \frac{1}{r} \frac{d}{dr} \right)^k \psi(r) \bigg|_{r=t}.\]

A bit of reflection, computing shows that
\[
\left( \frac{1}{r} \frac{d}{dr} \right)^k \psi(r) \bigg|_{r=t} = \left( \frac{1}{t} \frac{d}{dt} \right)^k \psi(t).
\]
so that we finally have what we wanted:

\[
(S_t, \varphi) = \sqrt{\frac{\pi}{2}} \left( \frac{1}{t} \frac{d}{dt} \right)^{n-3} \left( t^{n-2} \int_{S^{n-1}} \varphi(t\omega) \, dS_{\omega} \right). \tag{22}
\]

**Lemma 13** For \( t > 0 \), supp \( S_t = \{ x \in \mathbb{R}^n : |x| = t \} \).

### 2.5 Solution of the wave equation

Assuming that \( u \) was a solution of \( u_{tt} = c^2 \Delta u \) with initial values \( u(x,0) = 0 \), \( u_t(x,0) = g(x) \), we had (4),

\[
\hat{u}(\xi,t) = \sin c\frac{\xi}{t} \hat{g}(\xi).
\]

We will assume that \( g \in S(\mathbb{R}^n) \), then everything I am doing is perfectly justified. Then

\[
u(x,t) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \frac{\sin c|\xi| t}{c|\xi|} \hat{g}(\xi) \, d\xi. \tag{23}
\]

Keeping \( x \) fixed, \( \xi \mapsto e^{ix \cdot \xi} \hat{g}(\xi) \in S(\mathbb{R}^n) \) and it has Fourier transform

\[
(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i(y-x) \cdot \xi} \hat{g}(\xi) \, d\xi = g(x-y).
\]

In the last section, we assumed that In the notation of the last section, (23) can be written as

\[
u(x,t) = (2\pi)^{-\frac{n}{2}} \left\{ \frac{1}{c} S_{ct}, e^{ix \cdot \xi} \hat{g} \right\} = (2\pi)^{-\frac{n}{2}} \frac{1}{c} \langle S_{ct}, g(x-\cdot) \rangle.
\]

In using now (22), we have to consider that

\[
\frac{1}{ct} \frac{d}{dt} \frac{d}{d(ct)} = c^{-2} \frac{d}{d \frac{1}{t}} \frac{d}{d \left( \frac{1}{t} \right)}
\]

so that we finally (well, almost finally) get

\[
u(x,t) = (2\pi)^{-\frac{n}{2}} \sqrt{\frac{\pi}{2}} \left( \frac{1}{t} \frac{d}{dt} \right)^{\frac{n-3}{2}} \left\{ t^{n-2} \int_{S^{n-1}} g(x - ct\omega) \, dS_{\omega} \right\}. \tag{24}
\]

I switched to partial derivatives because now there is a second variable \( x \) present. There is no factor of \( c \) because

\[
\frac{1}{c} \left( \frac{1}{ct} \frac{d}{d(ct)} \right)^{\frac{n-3}{2}} = c^{-n+2} \left( \frac{1}{t} \frac{d}{dt} \right)^{\frac{n-3}{2}}
\]

and \( c^{-n+2} \) cancels with the \( c^{n-2} \) coming from replacing \( t^{n-2} \) by \( (ct)^{n-2} \) in (23).

For \( n = 3 \) we don’t even have to differentiate, the solution works out to

\[
u(x,t) = \frac{t}{4\pi} \int_{S^{n-1}} g(x - ct\omega) \, dS_{\omega}.
\]
In trying to keep the notation compact, let us define $W$ an operator from $S(R^n)$ to functions of $(x,t) \in R^n \times R$ by

$$Wg(x,t) = (2\pi)^{-\frac{n}{2}} \sqrt{\frac{n}{2}} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-1}{2}} \left\{ t^{n-2} \int_{S^{n-1}} g(x - ct\omega) dS_\omega \right\}. \quad (25)$$

Then the solution of the problem (1), which was

$$\begin{cases}
    u_{tt} = c^2 \Delta u,
    
    u(x,0) = f(x),
    
    u_t(x,0) = g(x),
\end{cases}$$

is given by

$$u(x,t) = \frac{\partial (Wf)}{\partial t}(x,t) + Wg(x,t). \quad (26)$$

THE END, for now