1 A few important partial differential equations

1.1 First, an important operator; the Laplacian

If $u = u(x,y,z)$ is a function of three variables, we define the Laplacian of $u$ by

$$\nabla^2 u = \text{div}(\text{grad } u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$ 

If $u = u(x,y)$ only depends on two variables, then the Laplacian is defined by

$$\nabla^2 u = \text{div}(\text{grad } u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$ 

And if $u = u(x)$ depends on a single variable, the Laplacian reduces to the second derivative $\frac{d^2 u}{dx^2}$ of $u$.

Convention. Frequently our functions will depend on one, two or three space variables, denoted by $x,y,z$ (or $x_1,x_2,x_3$) in the case of three space variables, and an extra variable $t$ interpreted as being time. The Laplacian will then be interpreted as acting only on the space variables. So if $u = u(x,y,t)$, then we still have

$$\nabla^2 u = \text{div}(\text{grad } u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$ 

1.2 The heat equation

Let $U$ be a region of space (one, two or three, or even more dimensional if one can visualize this) and let $u = u(x,y,z,t)$ be the temperature at the point $(x,y,z)$ at time $t$ in the three dimensional case; $u(x,y,t)$ the temperature at $(x,y)$ at time $t$ in the two dimensional case, $u(x,t)$ the temperature at $x$ at time $t$ in the one dimensional case.

A simple model (more on it below) states that $u$ satisfies an equation of the form

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

where $k$ is a constant. This is the heat equation.

1.3 The wave equation

A model for waves traveling in space is described by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u.$$ 

For example, $u = u(x,y,z,t)$ could be the pressure at point $(x,y,z)$, at time $t$ due to a vibration produced by a sound source. In this context, $c$ is the velocity of propagation of the wave.

1.4 Poisson and Laplace’s equations

Poisson’s equation is simply of the form

$$\nabla^2 u = f;$$

$f$ being a given function, $u$ the unknown. If $f = 0$ it is called Laplace’s equation. These equations govern stationary phenomena, like the distribution of an electric field or the temperature of a body once equilibrium has been reached.

Notes on notation: Instead of $\nabla^2$, one also writes $\Delta$ for the laplacian. It is the preferred notation of our textbook, so that’s what we’ll use from now on.
2 Some energy very basic basics

When a body of mass $m$ falls in a vacuum from a height $h$ subject to the force due to the acceleration $g$ of gravity its velocity is given by

$$\dot{v} = \frac{dv}{dt} = -g.$$  

If $x = 0$ is at ground level, then its height above the ground at time $t$ is given by $x = -\frac{1}{2}gt^2 + h$. But all we use is that $\dot{x} = v$ and

$$\frac{d}{dt} \left( \frac{1}{2}mv^2 + mgx \right) = mv\dot{v} + mgv = mv(\dot{v} + g) = 0.$$  

The quantity $E = \frac{1}{2}mv^2 + mgx$ remains constant. We call $E_k = \frac{1}{2}mv^2$ the kinetic energy of the system, $E_p = mgx$ is the potential energy. At time 0, $E_k = 0$ and $E_p = mgh$. On hitting the ground, assuming that then $x = 0$, we have $E_k$ quite large and $E_p = 0$. A basic principle of physics is the principle of conservation of energy. As the object falls, potential energy is transformed into kinetic energy, but the total energy $E$ remains constant.

However, if we have the same object fall in a non-vacuous environment (air, water, whatever), the velocity does not increase as much as the object is falling and $E_k + E_p$ fails to be constant. To keep the principle of conservation of energy intact, the British physicist (and brewer) James Prescott Joule posited a new form of energy, thermal energy.

![J.P. Joule (1819-1889)](Pictures copied from Wikipedia)

![Joule's heating apparatus](Pictures copied from Wikipedia)

In modern terms, thermal energy is energy due to random motions of the atoms of a substance and it is manifested by what we call heat and measured in terms of temperature. Heat and thermal energy are sometimes
3 DERIVATION OF THE HEAT EQUATION

used interchangeably, but thermal energy is intrinsic to a body; heat is the way thermal energy is transferred from one body to another. We might recall, or learn that the physics unit of work or energy known as joule is the work done by a force of one newton to move a mass of one kg a distance of one meter. As mentioned, the way we measure increases or decreases of thermal energy in a body is by its temperature. The specific heat of a substance is then defined as the amount of energy that has to be applied to raise the temperature of a unit mass of the substance by one unit of temperature. For example, it takes 4200 joules to raise the temperature of one kg of water by one degree celsius, thus the specific heat of water is

\[
\frac{4200 \text{ joules}}{\text{kg} \times \text{degrees celsius}}.
\]

3 Derivation of the heat equation

We consider a rod that has its sides totally insulated, so the only way it can receive or lose energy is at its endpoints. We also consider its cross sections are constant of area \(A\) and everything thermal is constant along any one cross section. We set up a coordinate axis with \(x = 0\) at one end-point of the rod. The other endpoint is then at \(x = L\). Let \(E(x, t)\) be the thermal energy density at point \(x\), at time \(t\). Concentrating on a portion of the rod between \(x\) and \(x + \Delta x\) (\(\Delta x > 0\) and small), the total energy in that portion of the rod is

\[
\int_{x}^{x+\Delta x} E(y, t) A \, dy.
\]

If this energy changes it can be due to two reasons (at least, these are the only two reasons we consider in this model):

1. **Heat flux.** It is observed that thermal energy, in the form of heat, goes from areas of high temperature to areas of lower temperature. We denote by \(\phi(x, t)\) the amount of thermal energy flowing to the right per unit time, per unit area at \(x\). So \(\phi > 0\) means the region loses energy, \(\phi < 0\) means energy is flowing in.

2. **Heat sources.** At some points in the rod there could be some device increasing or decreasing the thermal energy. We denote by \(q(x, t)\) the heat energy generated by unit volume, unit time, at \(x\) at time \(t\).

So concentrating on the portion of the rod between \(x\) and \(x + \Delta x\), time ranging from \(t\) to \(t + \Delta t\), we see that the rate of change in thermal energy is

\[
\frac{d}{dt} \int_{x}^{x+\Delta x} E(y, t) A \, dy = \int_{x}^{x+\Delta x} \frac{\partial E}{\partial t}(y, t) A \, dy.
\]

The rate of change due to heat flux consists in what goes out (or comes in) at \(x + \Delta x\), minus what goes out (or comes in) at \(x\). It is

\[-[\phi(x + \Delta x, t) A - \phi(x, t) A].\]

The reason for the negative sign was explained above.

The rate of change due to heat sources:

\[
\int_{x}^{x+\Delta x} q(y, t) A \, dx.
\]

Equating we get

\[
\int_{x}^{x+\Delta x} \frac{\partial E}{\partial t}(y, t) A \, dy = -[\phi(x + \Delta x, t) A - \phi(x, t) A] + \int_{x}^{x+\Delta x} q(y, t) A \, dx.
\]

If we divide by \(A\Delta x\) and let \(\Delta x \to 0\) we get

\[
\frac{\partial E}{\partial t} = -\frac{\partial \phi}{\partial x} + q. \tag{1}
\]

We now relate it all to temperature. This can be done via the specific heat \(c\). Since our rod might not be made out of a single material, let \(\rho(x)\) be the (linear) density at \(x\) so that the portion between \(x\) and \(x + \Delta x\) has mass \(\int_{x}^{x+\Delta x} \rho(y) A \, dy\). Then \(c = c(x)\), also can (and will) depend on \(x\). These quantities are related by the equation

\[
E(x, t) = c(x)\rho(x)u(x, t).
\]
In terms of temperature (1) becomes
\[ c \rho \frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + q. \]
(\( c, \rho \) pull out of \( \frac{\partial}{\partial t} \) since we assume they are independent of time.)

Enter Joseph Fourier.

Joseph Fourier (1768-1830)

Fourier, in his study of heat, posited that the flow was proportional to the temperature gradient. More precisely, in three dimensions, given a surface patch \( S \) of normal vector \( \mathbf{n} \), the heat flux across this surface is proportional to \( \nabla u \cdot \mathbf{n} \), where \( u \) is the temperature.

In our one dimensional case, taking \( S \) to be a cross section of the rod, it reduces to
\[ \phi = -K_0 \frac{\partial u}{\partial x}. \]
This is *Fourier’s law of heat conduction*. The constant $K_0$ is known as *thermal conductivity*. Since it depends on the material, in our one-dimensional case it could depend on $x$. Using Fourier’s law in (2) we get

$$c \rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( K_0 \frac{\partial u}{\partial x} \right) + q.$$  \hspace{1cm} (3)

For the case of a homogeneous medium without any heat sources, $K_0, c, \rho$ are all constant, $K_0$ pulls out of $\frac{\partial}{\partial x}$ and introducing $k = \frac{K_0}{c \rho}$ we can write the equation in the form

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}.$$  \hspace{1cm} (4)

It is quite important that $k > 0$.

If there are heat sources, the equation becomes

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + q.$$  \hspace{1cm} (5)