1. Textbook, Exercise 76, Section 3.5 (p. 217): Find the equations of both the tangent lines to the ellipse \( x^2 + 4y^2 = 36 \) that pass through the point \((12, 3)\).

Solution. The point is clearly NOT on the ellipse. We know that a line through \((12, 3)\) will have the form \( y = m(x - 12) \), so all we need to do is find the slope \( m \). The line will touch the ellipse at some point \((x_0, y_0)\). If we differentiate implicitly the equation defining the ellipse we get \( 2x + 8yy' = 0 \), thus \( y' = -\frac{x}{4y} \).

At the point \((x_0, y_0)\) the slope is \( \frac{x}{4y_0} \). So \( m = \frac{x}{4y_0} \). But since the point is on the line we have \( y_0 = \frac{3}{x_0/4y_0} \).

Multiplying by \( 4y_0 \), rearranging we get

\[
x_0^2 + 4y_0^2 = 12x_0 + 12y_0.
\]

Since the point has to be on the ellipse, we also have \( x_0^2 + 4y_0^2 = 36 \) so that \( 12x_0 + 12y_0 = 36 \). Solving for \( y_0 \), \( y_0 = 3 - x_0 \). Now some heavier lifting may be needed. Let’s substitute this into the equation for the ellipse:

\[
36 = x_0^2 + 4(3 - x_0)^2 = 5x_0^2 - 24x_0 + 36,
\]

or \( 5x_0^2 - 24x_0 = 0 \).

The last equation has the solutions \( x_0 = 0 \) and \( x_0 = \frac{24}{5} = \frac{2}{3}x - 5 \).

The two slopes are 0 and \((\frac{24}{5})/(-\frac{36}{5}) = \frac{2}{3} \). The lines are

\[
y = 3 \quad \text{and} \quad y = \frac{2}{3}x - 5.
\]

2. Textbook, Exercise 18, Section 3.7 (p. 234): If a tank holds 5000 gallons of water that drains from the bottom of the tank in 40 minutes, then Torricelli’s Law gives the volume \( V \) of water remaining in the tank after \( t \) minutes as

\[
V = 5000 \left(1 - \frac{1}{40}t\right)^2, \quad 0 \leq t \leq 40.
\]

Find the rate at which the water is draining from the tank after (a) 5 min. (b) 10 min. (c) 20 min. and (d) 40 min. At what time is water flowing out the fastest? The slowest? Summarize your results.

Solution. The rate at which the volume changes is given by \( dV/dt \). The rate at which it flows out is the exact opposite, namely \(-dV/dt\). We have

\[
-\frac{dV}{dt} = 250 \left(1 - \frac{1}{40}t\right) = 250 - \frac{25}{4}t.
\]

The asked for rates are

(a) For \( t = 5 \), the rate is \( \frac{875}{4} \) gallons per minute.

(b) For \( t = 10 \), the rate is \( \frac{375}{2} \) gallons per minute.

(c) For \( t = 20 \), the rate is 125 gallons per minute.

(d) For \( t = 40 \), the rate is 0 gallons per minute.

The rate is a linear function, the graph is a line of negative slope, so it is constantly going down. The rate of flow is thus fastest at the very first \((t = 0)\) and at the very end \((t = 40)\).
3. Textbook, Exercise 26, Section 3.7 (p. 235): The number of yeast cells in a laboratory culture increases rapidly initially but levels off eventually. The population is modeled by the function

\[ n = f(t) = \frac{a}{1 + be^{-0.7t}} \]

where \( t \) is measured in hours. At time \( t = 0 \) the population is 20 cells and is increasing at the rate of 12 cells/hour. Find the values of \( a \) and \( b \). According to this model, what happens to the yeast population in the long run?

**Solution.** The function given is actually what one gets from a relatively simple population model. The information given about what happens at time \( t = 0 \) tells us

\[ f(0) = 20; \quad \text{thus} \quad \frac{a}{1 + b} = 20; \quad \text{i.e.} \quad a = 20(1 + b); \]

and, since \( f'(t) = \frac{0.7abe^{-0.7t}}{(1 + be^{-0.7t})^2} \),

\[ f'(0) = 12; \quad \text{thus} \quad \frac{0.7ab}{(1 + b)^2} = 12; \quad \text{so} \quad 0.7ab = 12(1 + b)^2. \]

We have two equations for \( a, b \), which are easily solvable to give \[ a = 140, b = 6. \] To find out what the population does in the long run, we take the limit of \( f(t) \) as \( t \to \infty \):

\[ \lim_{t \to \infty} f(t) = \frac{a}{1 + b} \lim_{t \to \infty} e^{-0.7t} = \frac{a}{1 + b} \cdot 0 = a. \]

In the long run, the population gets closer and closer to \( a = 140 \), without ever passing it.

4. Textbook, Exercise 38, Section 3.9 (p. 250). I refer to the textbook for the statement of the problem.

**Solution.** The theorem of Pythagoras is the main tool here. Let \( x(t) \) be the distance of cart \( A \) from \( Q \), and let \( y(t) \) be the distance of cart \( B \) from \( Q \). Notice that doing it this way, \( x(t) \) increases, thus its derivative is positive, while \( y(t) \) decreases. The velocity of cart \( B \) to \( Q \) is going to be \( -y'(t) \). An application of Pythagoras tells us that

\[ \sqrt{x(t)^2 + 144} + \sqrt{y(t)^2 + 144} = 39; \quad (1) \]

differentiating with respect to \( t \):

\[ \frac{x(t)x'(t)}{\sqrt{x(t)^2 + 144}} + \frac{y(t)y'(t)}{\sqrt{y(t)^2 + 144}} = 0. \quad (2) \]

When cart \( A \) is 5 ft. from \( Q \), we get from equation \( (1) \) (plugging in \( x = 5 \), solving for \( y \)) that \( y = \sqrt{532} = 2\sqrt{133} \). If we use this value of \( y \), \( x = 5 \) and \( x' = 2 \) in equation \( (2) \), it becomes a simple equation for \( y' \) that solves to \( y' = -10/\sqrt{133} \).

Cart B is moving toward \( Q \) at a speed of \( \frac{10}{\sqrt{133}} \) feet per second.