Calculus with Analytic Geometry I
Sample Final Exam/Review Questions
Solutions

The instructions for the final exam will be as usual: **SHOW ALL WORK. WRITE CLEARLY AND IN FULL SENTENCES. IMPROPER USE OF THE EQUAL SIGN WILL BE PENALIZED. NOT USING AN EQUAL SIGN WHERE IT SHOULD BE USED WILL ALSO BE PENALIZED. USE EXTRA PAPER AS NEEDED. IN OPTIMIZATION PROBLEMS, YOU MUST JUSTIFY THAT A MAXIMUM OR MINIMUM HAS BEEN ACHIEVED.**

The final exam will definitely be shorter, and probably easier, so don’t panic. I want to partially use these questions to review a lot of what we covered.

1. Compute the following limits. Use any procedure you want, but show ALL work.

   (a) \( \lim_{x \to 0} \frac{x - \tan x}{x^3} \).
   
   **Solution.** It is an indeterminacy of the type 0/0. L'Hôpital applies.
   
   \[
   \lim_{x \to 0} \frac{x - \tan x}{x^3} = \lim_{x \to 0} \frac{1 - \sec^2 x}{3x^2} = \lim_{x \to 0} \frac{-2 \sec^2 x \tan x}{6x} = \lim_{x \to 0} \frac{-2 \sec^2 x}{6} = -\frac{2}{6} = -\frac{1}{3}
   \]

   (b) \( \lim_{x \to 3} \frac{x^3 - 3x^2 + 2x - 6}{x^2 - 9} \).
   
   **Solution.** An indeterminacy of the type 0/0. By L'Hôpital,
   
   \[
   \lim_{x \to 3} \frac{x^3 - 3x^2 + 2x - 6}{x^2 - 9} = \lim_{x \to 3} \frac{3x^2 - 6x + 2}{2x} = \frac{11}{6}.
   \]

   (c) \( \lim_{x \to \infty} \frac{(x^2 + 1)^3}{(x - 1)^2(2x + 1)^4} \).
   
   **Solution.** It is an indeterminacy of the form \( \infty/\infty \). But rather than applying L'Hôpital, we notice that multiplied out, the numerator is a polynomial of the form \( x^6 + \) lower order terms, the denominator of the form \( 16x^6 + \) lower order terms, thus
   
   \[
   \lim_{x \to \infty} \frac{(x^2 + 1)^3}{(x - 1)^2(2x + 1)^4} = \frac{1}{16}.
   \]

   (d) \( \lim_{x \to 0} (1 + 3x)^{1/x} \).
   
   **Solution.** Indeterminacy of the form \( 1^\infty \). Let \( y = (1 + 3x)^{1/x} \). Then \( \ln y = \ln(1 + 3x)/x \) has an indeterminacy of the form 0/0 as \( x \to 0 \). By L'Hôpital,
   
   \[
   \lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1 + 3x)}{x} = \lim_{x \to 0} \frac{3}{1 + 3x} = 3.
   \]

   Thus
   
   \[
   \lim_{x \to 0} (1 + 3x)^{1/x} = \lim_{x \to 0} e^{\ln y} = e^{\lim_{x \to 0} \ln y} = e^3.
   \]

2. Evaluate the following derivatives. Since the assumption is that you did lot of exercises and now are at least as good as any calculator in finding derivatives, it is OK to just write the answer.
(a) \( y = \sin \left( \frac{1}{x^2 + 1} \right) \), find \( \frac{dy}{dx} \).

Solution.

\[
\frac{dy}{dx} = -\sin \left( \frac{1}{x^2 + 1} \right) \frac{2x}{(x^2 + 1)^2}.
\]

(b) \( f(x) = 2^x \sin x + \frac{x \sin x}{2x + 1} \), find \( f'(x) \).

Solution.

We might want to compute things separately. First we compute \( \frac{d(2^x \sin x)}{dx} \). Let \( y = 2^x \sin x \), then

\[
\ln y = (x \sin x) \ln 2; \ \ \text{differentiating,} \ \ \frac{y'}{y} = (x \cos x + \sin x) \ln 2 \ \ \text{and}
\]

\[
\frac{d(2^x \sin x)}{dx} = y' = (x \cos x + \sin x)(\ln 2)2^x \sin x.
\]

Next,

\[
\frac{d}{dx} \frac{x \sin x}{2x + 1} = \frac{(x \cos x + \sin x)(2x + 1) - 2x \sin x}{(2x + 1)^2} = \frac{(2x + 1)x \cos x + \sin x}{(2x + 1)^2}.
\]

Thus

\[
f'(x) = (x \cos x + \sin x)(\ln 2)2^x \sin x + \frac{(2x + 1)x \cos x + \sin x}{(2x + 1)^2}.
\]

(c) \( g(x) = x \cos x \), find \( g'(x), g''(x), g'''(x) \).

Solution.

\[
g'(x) = -x \sin x + \cos x
\]

\[
g''(x) = -x \cos x - 2 \sin x
\]

\[
g'''(x) = x \sin x - 3 \cos x
\]

(d) \( g(x) = x \cos x \), find a general formula for the \( n \)-th derivative.

Solution. A pattern emerges. It works out to

\[
g^{(n)}(x) = \begin{cases} \ (-1)^k (x \cos x + n \sin x) & \text{if } n = 2k \text{ is even,} \\ \ (-1)^{k+1} (x \sin x - n \sin x) & \text{if } n = 2k + 1 \text{ is odd.} \end{cases}
\]

3. Find the tangent line to the curve of equation \( x^4 - 6xy^2 + y^4 = 25 \) at the point (3, 2).

Solution. This is an implicit function problem. Differentiating implicitly,

\[
4x^3 - 6y^2 - 12xyy' + 4y^3y' = 0.
\]

Solving for \( y \),

\[
y' = \frac{6y^2 - 4x^3}{4y^3 - 12xy}.
\]

Evaluating at (3, 2) we get \( y' = 21/10 = 2.1 \). That’s the slope. The equation of the line is thus \( y - 2 = 2.1(x - 3) \) or \( y = 2.1x - 4.3 \). Here’s a picture.
4. For the following curves:

(a) Determine ALL horizontal and vertical asymptotes.
(b) Determine the intervals of increase and decrease.
(c) Find all critical points and classify them as relative maximum, minimum, or neither.
(d) Determine the intervals of concavity.
(e) Find all inflection points.
(f) Use the preceding information to sketch a graph of the curve. If the graph is inconsistent with your information you will get NO credit for the whole exercise.

(a) \( y = x^3 - 12x^2 + 36x \).

Solution. There are no vertical nor horizontal asymptotes. We have

\[
\frac{dy}{dx} = 3x^2 - 24x + 36.
\]
\[
\frac{d^2y}{dx^2} = 6x - 24.
\]

Setting the first derivative to 0, and simplifying, results in the quadratic equation \( x^2 - 8x + 12 = 0 \), which has the solutions \( x = 2, 6 \). These are critical points. Now

\[
\frac{d^2y}{dx^2} \bigg|_{x=2} = -12 < 0, \quad \frac{d^2y}{dx^2} \bigg|_{x=6} = 12 > 0.
\]

Thus there is a local maximum at 2, a local minimum at 6. Setting the second derivative to 0 gives us an equation whose only solution is \( x = 4 \). Notice that our investigation already tells us that it is an inflection point, because at 2 the second derivative is negative, at 6 it is positive. The second derivative changes sign at 4. We try for some intercepts. 0 is both an \( x \) and \( y \) intercept. The other \( x \) intercepts can be found by solving \( x^2 - 12x + 36 = 0 \). There is only one solution, \( x = 6 \), also a critical point. We are ready to answer the questions.

i. Determine ALL horizontal and vertical asymptotes. There are none.
ii. Determine the intervals of increase and decrease.
   The function increases in \((-\infty, 2)\) and in \((6, \infty)\).
iii. Find all critical points and classify them as relative maximum, minimum, or neither.
   The critical points are \( x = 2 \), where we have a local maximum, and \( x = 6 \); a local minimum.
iv. Determine the intervals of concavity.
   The function is concave up in (4, ∞), down in (−∞, 4).

v. Find all inflection points.
   There is an inflection at x = 4.

Finally, the plot. The points we plot are:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>x and y-intercepts</td>
</tr>
<tr>
<td>2</td>
<td>32</td>
<td>local max</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>inflection</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>local min and x-intercept</td>
</tr>
</tbody>
</table>

We plot these points, drawing short tangents at the critical points.

Then we fill in as well we can, consistent with our information. The graph looks like:
(b)  \( y = \frac{\sin x}{2 + \cos x}, \quad -\pi \leq x \leq \pi. \)

**Solution.**  This time I’ll write \( f(x) = \frac{\sin x}{2 + \cos x} \). The denominator is always positive; \( 2 + \cos x > 0 \) for all \( x \). The function is an odd function, so the graph for \( -\pi \leq x \leq 0 \) can be obtained from that for \( 0 \leq x \leq \pi \) by reflecting across the \( x \)-axis, and then across the \( y \)-axis. I won’t use this symmetry too much, except to keep it in mind.

I begin computing first and second derivatives.

\[
 f'(x) = \frac{\cos x(2 + \cos x) + \sin^2 x}{(2 + \cos x)^2} = \frac{2 \cos x + \cos^2 x + \sin^2 x}{(2 + \cos x)^2} = \frac{2 \cos x + 1}{(2 + \cos x)^2}. 
\]

The second derivative works out to

\[
 f''(x) = \frac{(-2 \sin x)(2 + \cos x) - (2 + \cos x)(-\sin x)(2 \cos x + 1)}{(2 + \cos x)^4} 
\]

This looks pretty grim, but one can cancel a factor of \( (2 + \cos x) \) from the top, and it simplifies to

\[
 f''(x) = -\frac{2 \sin x(1 - \cos x)}{(2 + \cos x)^3}. 
\]
Critical points are solutions of $2 \cos x + 1 = 0$; that is $\cos x = -1/2$. Every value but the value 1 is assumed twice by cosine in $[-\pi, \pi]$ so we have two critical points: namely $-2\pi/3$ and $2\pi/3$. Oddness of the original function tells us that if one is a local max, the other one must be a local min, and vice-versa. We can verify this at once using the second derivative. Looking at the expression for $f''(x)$, since $1 - \cos x \geq 0$ and the denominator is $> 0$, the sign of $f''$ at the critical points is completely determined by that of the sine, and (because of the minus in front) is opposite to that of sine. We conclude that $f''(-2\pi/3) > 0$ (a local minimum), $f''(2\pi/3) < 0$ (a local maximum).

Potential inflection points will be solutions of $\sin x(1 - \cos x) = 0$. The solutions are $\pm \pi$ and 0. $\pm \pi$ are the endpoints of the interval, but the function is defined for all $x$, so we could have inflections at the end-points. Something to keep in mind. Because $f''(-2\pi/3) > 0$ and $f''(2\pi/3) < 0$, we conclude that 0 is in fact an inflection point. So is $\pi$. Because the function is periodic of period $2\pi$, the behavior to the right of $\pi$ is the same as to the right of $-\pi$, so $f''$ again becomes positive past $\pi$. Similarly $-\pi$ is an inflection point. To conclude our analysis, even though we are not asked about it, the $x$-intercepts are $-\pi, 0, \pi$, the $y$ intercept is 0. We are ready to answer the questions.

i. Determine ALL horizontal and vertical asymptotes.

There is no vertical asymptote; since the interval of definition is finite, the question about horizontal asymptotes does not apply.

ii. Determine the intervals of increase and decrease.

Because there are only two critical points, the first one a local min, the second one a local max, it is clear (or should be) that the function is decreasing in $(-\pi, -2\pi/3)$ and $(2\pi/3, \pi)$; increasing in $(-2\pi/3, 2\pi/3)$.

iii. Find all critical points and classify them as relative maximum, minimum, or neither.

The critical points are $-2\pi/3$, a local minimum, and $2\pi/3$, a local maximum.

iv. Determine the intervals of concavity.

The function is concave up in $(-\pi, 0)$ (where the local min occurs); it is concave down in $(0, \pi)$.

v. Find all inflection points.

$-\pi, 0, \pi$ are the inflection points

vi. Use the preceding information to sketch a graph of the curve.

We start the plot by plotting all the points we found so far. At the critical points we draw a short horizontal line to remind us they are critical and the tangent should be horizontal there. We also draw short tangential segments at the inflection points so as to better thread the graph through those points. Points to plot:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\pi$</td>
<td>0</td>
<td>$x$-intercept, inflection, $y' = -1$</td>
</tr>
<tr>
<td>$-2\pi/3 \approx -2.094$</td>
<td>$-1/\sqrt{3} \approx -0.577$</td>
<td>local min</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>$x$, $y$-intercept, inflection, $y' = 1/3$</td>
</tr>
<tr>
<td>$2\pi/3 \approx 2.094$</td>
<td>$1/\sqrt{3} \approx 0.577$</td>
<td>local min</td>
</tr>
<tr>
<td>$\pi$</td>
<td>0</td>
<td>$x$-intercept, inflection, $y' = -1$</td>
</tr>
</tbody>
</table>

What we get, if we do this (as we should) will look like

![Graph Image]

It is now a breeze to complete the plot:
(c) \( y = \frac{x}{x^3 - 1} \).

Solution. The first thing to notice is that this function has a problem at \( x = 1 \); the line \( x = 1 \) is a vertical asymptote. While not a critical point because it isn’t in the domain, concavity and whether the function increases or decrease can change at 1. I will write the simplified form of the first and second derivatives. They work out to:

\[
\frac{dy}{dx} = -\frac{2x^3 + 1}{(x^3 - 1)^2},
\]

\[
\frac{d^2y}{dx^2} = \frac{6x^2(x^3 + 2)}{(x^3 - 1)^3}.
\]

There is only one critical point, namely \( x = -\frac{1}{\sqrt[3]{2}} \). It is easy to see it is a local maximum. For example, if we plug it into \( y'' \), the numerator is positive, the denominator negative; thus \( y''(−\frac{1}{\sqrt[3]{2}}) < 0 \). We can also see that \( y' \) goes from positive to negative as we pass the critical point.

Concerning the second derivative, it is 0 for \( x = -\frac{3}{\sqrt{2}} \) and 0. We see that the numerator of \( y'' \) will be negative for very negative values of \( x \); the denominator is negative as long as \( x < 1 \). Thus \( y'' > 0 \) for large negative values of \( x \). At the critical point, which is to the right of \( -\frac{1}{\sqrt[3]{2}} \) but to the left of 0, \( y'' < 0 \); thus we have an inflection. There is no sign change at 0, so 0 is not an inflection. Evaluating \( y'' \) at some point to the right of 1; say at 2, we get \( y''(2) > 0 \), so \( y'' > 0 \) to the right of 1. The only \( x \)-intercept is 0; 0 is also the \( y \)-intercept.

We are ready to answer the questions.

i. Determine ALL horizontal and vertical asymptotes.

\( x = 1 \) is a vertical asymptote and, because \( \lim_{x \to \pm \infty} \frac{x}{(x^3 - 1)} = 0 \), \( y = 0 \) is asymptotic both at \( -\infty \) and at \( \infty \).

ii. Determine the intervals of increase and decrease.

Because the only critical point \( -\frac{1}{\sqrt[3]{2}} \) is a local maximum, the function **increases** in \( (-\infty, -\frac{1}{\sqrt[3]{2}}) \), **decreases** in \( (-\frac{1}{\sqrt[3]{2}}, 1) \). To know what happens past 1 we evaluate the derivative at any point, say at 2. We get \( y'(2) = -17/49 \), so it **decreases** also in \( (1, \infty) \). We could also have deduced this from the fact that it is concave up (remember \( y'' \) was positive) and asymptotic to the \( x \)-axis.

iii. Find all critical points and classify them as relative maximum, minimum, or neither.

There is one relative maximum \( -\frac{1}{\sqrt[3]{2}} \).

iv. Determine the intervals of concavity.

The function is **concave up** in \( (-\infty, -\sqrt[3]{2}) \) and in \( (1, \infty) \), **concave down** in \( (-\sqrt[3]{2}, 1) \)

v. Find all inflection points.

\( -\sqrt[3]{2} \) is the only point where an inflection occurs.

vi. Use the preceding information to sketch a graph of the curve.

Points to plot:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( -\sqrt[3]{2} \approx -1.26 )</td>
<td>( \sqrt[3]{2}/3 \approx 0.42 )</td>
<td>inflection, ( y' = 1/3 )</td>
</tr>
<tr>
<td>( -1/\sqrt[3]{2} \approx -0.79 )</td>
<td>( \sqrt[3]{4}/3 \approx 0.53 )</td>
<td>local max</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>( x, y )-intercept</td>
</tr>
</tbody>
</table>

We plot these points, draw short tangents as usual, and draw the asymptotes. The horizontal asymptote is the \( x \)-axis, so it won’t really show. The vertical asymptote is the vertical red line.
Now we can easily complete the graph. To the right of the red line the curve is concave up and approaches the vertical asymptote and the x-axis as \( x \to \infty \). That doesn’t give us too many choices. We might want to add a plot point to get the graph better into position, say the point \((2, 2/7)\). To the left of \( x = 1 \), make sure it approaches the x-axis as \( x \to -\infty \) and the red line as \( x \to 1^- \).

It works out to

5. (Exercise 4.7# 72 of the textbook) A rain gutter is to be constructed from a metal sheet of width 30 cm bending up one third of the sheet on each side through an angle \( \theta \). How should \( \theta \) be chosen so that the gutter will carry the maximum amount of water?

Solution. What we need to do is to maximize the area of the cross-section. The cross section is a trapezoid.
The area of a trapezoid is the half-sum of its bases times the height. Its lower base is 10 cm long. The upper "base" can be evaluated with a bit of trigonometry as being 10 + 2 × 10 cos θ = 10 + 20 cos θ cm long. The height is also easily computed as h = 10 sin θ. The area of the cross-section, as a function of θ, is

\[ A(θ) = \frac{1}{2}(10 + 10 + 20 \cos θ) \times 10 \sin θ = 100(\sin θ + \cos θ \sin θ). \]

The domain for this function is 0 ≤ θ ≤ π/2. It is a continuous function of θ, so it must assume a maximum value somewhere in its domain, because the domain is a closed and bounded interval. The maximum must happen at an end-point, or at a critical point. To see if there are critical points, since A is everywhere differentiable, we find the derivative and set it to 0. We get

\[ A'(θ) = 100(\cos θ + \cos^2 θ - \sin^2 θ). \]

Setting this to 0 we get an equation involving a cosine, its square, and the square of the sine. A bit of thinking might tell us that it would be a good idea to replace the square of the sine by 1 minus the square of the cosine, so that

\[ A'(θ) = 100(\cos θ + 2 \cos^2 θ - 1). \]

Setting this to 0 gives

\[ 2 \cos^2 θ + \cos θ - 1 = 0, \]

a quadratic equation in \( \cos θ \). By the quadratic formula, \( \cos θ = \frac{-1 \pm \sqrt{1 + 8}}{4} \). The two solutions are \( \cos θ = 1/2 \) and \( \cos θ = -1 \). The second solution is outside of the domain, so we discard it. The first solution is \( θ = π/3 \) (60 degrees). We now compute

\[ A(0) = 0, \quad A(π/3) = 75\sqrt{3} ≈ 129.90, \quad A(π/2) = 100. \]

This shows the maximum is the value at \( π/3 \). The answer to the problem is \( θ = π/3 \).

6. (Almost the same as 4.7, # 49) An oil refinery is located on the north bank of a straight river that flows east to west. The river is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 km east of the the refinery. The cost of laying pipe is $400,000/km over land to a point \( P \) on the north bank and $750,000/km under the river to the tanks. To minimize the cost of the pipeline, where should \( P \) be located?

Solution. First of all, a picture.

In the picture, the river is blue, the oil refinery is the brown circle, the storage tanks the yellow rectangle. The pipeline is in red. The point \( P \) is at distance \( x \) from the refinery.

I’ll use as a monetary unit (mu) a hundred thousand dollars so that, for example, the cost of laying pipe over land is 4 mu/km. Then the cost of laying the pipe from the refinery to \( P \), and then straight across the river is, by Pythagoras

\[ C(x) = 4x + 7.5\sqrt{2^2 + (6 - x)^2} = 4x + 7.5\sqrt{x^2 - 12x + 40}. \]

*In doing this exercise in class, I made the mistake of assuming that all sections of the gutter were 30 cm long. The problem is essentially the same that way, the actual length of each section is not going to make a difference in the maximizing angle.
The domain of this function is \(0 \leq x \leq 6\). To determine critical points we differentiate and set the derivative to 0. We get
\[
C'(x) = 4 + \frac{7.5}{2}(x^2 - 12x + 40)^{-1/2}(2x - 12) = 4 + \frac{7.5(x - 6)(x^2 - 12x + 40)^{-1/2}}{2}.
\]
Setting equal to 0, and doing some algebra, we get
\[
-4(x^2 - 12x + 40)^{1/2} = 7.5(x - 6).
\]
Squaring (and keeping in mind that on squaring we lose the sign, which means some solutions we obtain could be meaningless)
\[
16(x^2 - 12x + 40) = 56.25(x^2 - 12x + 36).
\]
This can be rearranged into the quadratic equation
\[
40.25x^2 - 483x + 1385 = 0.
\]
This equation is quite ugly, but in this day and age of calculators not too terrifying, I hope. The solutions are
\[
x = \frac{483 \pm \sqrt{10304}}{80.5} = \frac{483 \pm 8\sqrt{161}}{80.5} = 6 \pm \frac{8\sqrt{161}}{80.5} \quad \text{or} \quad \frac{12}{80.5}.
\]
The only solution in the domain is \(6 - 8/\sqrt{161}/80.5 \approx 4.739\). We have a single critical point in the domain. **We need to justify that we found a minimum.** Several justifications are possible (and acceptable).

**Justification 1.** Because the domain is a closed and bounded interval, and \(C\) is continuous, \(C\) assumes a minimum value in the interval. This minimum value has to occur at one of the endpoints or at a critical point. Evaluating \(C\) at the critical points and endpoints, wherever the value is smallest, that has to be the minimum. We see
\[
C(0) \approx 47.43\text{ mu}, \quad C\left(6 - \frac{8\sqrt{161}}{80.5}\right) \approx 36.69\text{ mu}, \quad C(6) = 39\text{mu}.
\]
This justifies that the value at the \(x\) we found is the minimum.

**Justification 2.** Since \(x = 6 - \frac{8\sqrt{161}}{80.5}\) is the unique critical point, it will produce the minimum value if it is a local minimum. Notice that \(C'(0) = 4 - [45/\sqrt{40}] \approx -3.12 < 0\), \(C'(6) = 4 > 0\); by the first derivative test there is a local minimum at the critical point; by the single critical point criterion it is a global minimum.

7. (Exercise 4.7, # 16) A rectangular storage container with an open top is to have a volume of 10m³. The length of the base is twice the width. Material for the base costs $10 per square meter. Material for the sides costs $6 per square meter. Find the cost of materials for the cheapest such container.

**Solution.** I will omit a picture, but you should draw one. Let \(x\) be the width of the base, then \(2x\) is the length. Let \(h\) be the height. The container has a base of area \(2x \times x = 2x^2\text{ m}^2\); it has two sides of area \(2x \times h = 2xh\text{ m}^2\), two sides of area \(xh\text{ m}^2\). The cost of building it is thus
\[
C = 10 \times 2x^2 + 6 \times (4xh + 2xh) = 20x^2 + 36xh
\]
dollars. The constraint is that the volume is 10 cubic meters, in other words, \(2x^2h = 10\) or \(h = 5/x^2\). Replacing in the expression for \(C\) we get
\[
C(x) = 20x^2 + \frac{180}{x}.
\]
The domain is \(0 < x < \infty\). To find critical points we differentiate \(C\) and set to 0, we get \(40x - (180/x^2) = 0\), which solves to \(x = \sqrt[3]{4.5}\). We need to decide that \(C(\sqrt[3]{4.5})\) is the minimum value. Here are two acceptable justifications.

**Justification 1.** The second derivative is easy to compute; it works out to \(C''(x) = 40 + (360/x^3)\) and clearly \(C''(\sqrt[3]{4.5}) > 0\). Thus we have a local minimum; being the only critical point the value at that point is the minimum value.
Justification 2. Notice that \( \lim_{x \to 0} C(x) = \infty = \lim_{x \to \infty} C(x) \). This implies that \( C \) must assume a minimum value; since it has to assume it at a critical point and there is only one critical point, the value it assumes there must be the minimum.

The question was about the cost of the materials. It is
\[
C(\sqrt[3]{4.5}) = 20(4.5)^{2/3} + \frac{180}{4.5^{1/3}} = \frac{20 \cdot 4.5 + 180}{4.5^{1/3}} = 60(4.5)^{2/3} \approx 163.54
\]
dollars.

8. An object moves along a line. If its acceleration is given by \( a(t) = \sin(\pi t) \) (in ft/s\(^2\), \( t \) being measured in seconds), and its initial velocity is 3 ft/s.

(a) find its velocity function \( v(t) \)

(b) If at time \( t = 0 \) it is at the origin \((s(0) = 0)\), find its position at time \( t = 2 \) and the total distance traveled.

Solution. \( v(t) = \int a(t) \, dt = \int \sin(\pi t) \, dt \). To compute this integral we can make the substitution \( u = \pi t \), \( du = \pi \, dt \), thus
\[
\int \sin(\pi t) \, dt = \frac{1}{\pi} \int \sin u \, du = -\frac{1}{\pi} \cos u + C = -\frac{1}{\pi} \cos(\pi t) + C.
\]
Thus \( v(t) = -\frac{1}{\pi} \cos(\pi t) + C \). To determine \( C \) we set \( t = 0 \); use \( v(0) = 3 \); we get \( 3 = -\frac{1}{\pi} + C \), \( C = (3 + \pi)/\pi \).

The velocity function is
\[
v(t) = -\frac{1}{\pi} \cos(\pi t) + \frac{3 + \pi}{\pi}.
\]

To get the position function we have to integrate once more. We get \( s(t) = -\frac{1}{\pi^2} \sin(\pi t) + \frac{3 + \pi}{\pi} t + C \). Setting \( s = 0, t = 0 \), we get \( C = 0 \); thus the position function is
\[
\frac{3 + \pi}{\pi} t.
\]

At time 2, \( s(2) = \frac{2(3 + \pi)}{\pi} \).

To determine the distance travelled, on the other hand, we must integrate \( \int_0^2 |v(t)| \, dt \). To do this we have to determine where the velocity is positive, where negative. To determine where \( v(t) > 0 \), where \( v(t) < 0 \), we find out first where it is 0. If we are lucky, it won’t happen at all. We need to solve
\[
\frac{1}{\pi} \cos(\pi t) = \frac{3 + \pi}{\pi}; \quad \text{that is } \cos(\pi t) = 3 + \pi.
\]
The last equation has no solution, so \( v \) never changes sign. It is easy to see that \( v(t) > 0 \). It follows that the distance travelled is the same as the change in position, namely 2.

9. The height of a circular cone is increasing at the rate of 10 cm/s and the radius of its base at the rate of 5 cm/s. When the height is 80 cm and the radius of the base is 30 cm, how fast is the volume of the cone increasing. Recall that the formula for volume of a cone is \( V = \frac{1}{3} \pi r^2 h \).

Solution. We have \( V = \frac{1}{3} \pi r^2 h \) so that
\[
\frac{dV}{dt} = \frac{1}{3} \pi \left( 2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right).
\]
We are given that \( dr/dt = 5 \) and \( dh/dt = 10 \) (cm/s). Plugging all this into the formula for the change in rate of the volume, as well as \( r = 30 \) and \( h = 80 \), we get
\[
\frac{dV}{dt} = \frac{1}{3} \pi \left( 2 \times 30 \times 80 \times 5 + 30^2 \times 10 \right) = 11000 \pi
\]
cubic centimeters per second.
10. Of the following formulas, 2 are true, 2 are false. Decide which are true, which false, and justify your answer!

(a) \[ \int e^{x^2} \, dx = e^{x^2} + C. \]

Solution.

\[ \frac{d}{dx} e^{x^2} = 2xe^{x^2}. \]

There is no chance that \( 2xe^{x^2} = e^{x^2} \) (one function is odd, the other one is even, for example). Thus the formula is FALSE.

(b) \[ \int \frac{1}{1 + x^2} \, dx = \frac{1}{2} \left( \frac{x}{1 + x^2} + \arctan x \right) + C. \]

Solution.

\[ \frac{d}{dx} \left( \frac{1}{2} \left( \frac{x}{1 + x^2} + \arctan x \right) \right) = \frac{1}{2} \left( \frac{1 + x^2 - 2x^2}{(1 + x^2)^2} + \frac{1}{1 + x^2} \right) = \frac{1}{2} \left( \frac{1 - x^2}{(1 + x^2)^2} + \frac{1}{1 + x^2} \right) = \frac{1}{(1 + x^2)^2}. \]

The formula is TRUE.

(c) \[ \int \sin^3 x \, dx = \frac{1}{3} \cos^3 x - \cos x + C. \]

Solution.

\[ \frac{d}{dx} \left( \frac{1}{3} \cos^3 x - \cos x \right) = -\cos^2 x \sin x + \sin x = \sin x(1 - \cos^2 x) = \sin x \sin^2 x = \sin^3 x. \]

The formula is TRUE.

(d) \[ \int \frac{1}{1 + x^4} \, dx = \ln(1 + x^4) + C. \]

Solution.

\[ \frac{d}{dx} (\ln(1 + x^4)) = \frac{4x^3}{1 + x^4} \neq \frac{1}{1 + x^4}. \]

The formula is FALSE.

11. Compute the following integrals.

(a) \[ \int_0^2 x^2 \left( \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}} \right)^2 \, dx. \]

Solution.

\[ \int_0^2 x^2 \left( \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}} \right)^2 \, dx = \int_0^2 x^2 \left( x^{1/3} + x^{-1/3} \right)^2 \, dx = \int_0^2 x^2 \left( x^{1/3} + x^{-1/3} \right)^2 \, dx \]

\[ = \int_0^2 x^2 \left( x^{2/3} + 2 + x^{-2/3} \right) \, dx = \int_0^2 x^2 \left( x^{2/3} + 2 + x^{-2/3} \right) \, dx \]

\[ = \int_0^2 \left( x^{8/3} + 2x^{2/3} + x^{4/3} \right) \, dx = \frac{3}{11} x^{11/3} + \frac{2}{3} x^{3} + \frac{3}{7} x^{7/3} \bigg|_0^2 \]

\[ = \frac{3 \cdot 2^{11/3}}{11} + \frac{16}{3} + \frac{3 \cdot 2^{7/3}}{7}. \]

It could be simplified a bit more probably.

(b) \[ \int_0^3 \frac{2x^3}{x^4 + 5} \, dx. \]

Solution.
We substitute \( u = x^4 + 5 \), \( du = 4x^3 \, dx \). When \( x = 0 \), \( u = 5 \). When \( x = 3 \), \( u = 86 \). Thus
\[
\int_0^3 \frac{2x^3}{x^4 + 5} \, dx = \int_5^{86} \frac{1}{4} \frac{du}{u} = \frac{1}{2} \ln |u| \bigg|_5^{86} = \frac{1}{2} (\ln 86 - \ln 5) = \frac{1}{2} \ln \frac{86}{5}.
\]

(c) \( \int \frac{e^x}{1 + e^{2x}} \, dx \).

**Solution.** Substituting \( u = e^x \), we have \( du = e^x \, dx \), \( e^{2x} = u^2 \), thus
\[
\int \frac{e^x}{1 + e^{2x}} \, dx = \int \frac{1}{1 + u^2} \, du = \arctan u + C = \arctan(e^x) + C.
\]

(d) \( \int (x^4 + 1)^6 x^3 \, dx \).

**Solution.** Substitute \( u = x^4 + 1 \), \( du = 4x^3 \, dx \).
\[
\int (x^4 + 1)^6 x^3 \, dx = \frac{4}{1} \frac{1}{u^6} \, du = \frac{1}{28} u^7 + C = \frac{1}{28} (x^4 + 1)^7 + C.
\]

12. Find the derivative of the following functions

(a) \( g(x) = \int_1^{\sin x} \frac{1 - t^2}{1 + t^4} \, dt \).

**Solution.** \( g'(x) = \frac{1 - \sin^2 x}{1 + \sin^4 x} \cos x = \frac{\cos^3 x}{1 + \sin^4 x} \).

(b) \( h(x) = \int_\sqrt{x}^{3x^2} \frac{e^t}{t} \, dt \).

**Solution.** \( h'(x) = \frac{6xe^{3x^2}}{3x^2} - \frac{2e^{\sqrt{x}}}{\sqrt{x}} \).

13. Show that the equation \( 2x - \cos x = 0 \) has exactly one real root.

**Solution.** Let \( f(x) = 2x - \cos x \). A root of the equation is the same as a zero of \( f \). We see that \( f(0) = -1 < 0 \), while \( f(1) = 2 - \cos 1 > 0 \). By the intermediate value theorem, since \( f \) is continuous, there is some \( c, 0 < c < 1 \), such that \( f(c) = 0 \). This shows the equation has at least one root. If the equation were to have a second root, say \( d \), where \( d \neq c \), then \( f(c) = f(d) = 0 \). By Rolle’s theorem, there is \( a \) between \( c \) and \( d \) such that \( f'(a) = 0 \). But \( f'(x) = 2 + \sin x \) is never 0. Thus no such second root can exist.