The instructions for the final exam will be as usual: **SHOW ALL WORK. WRITE CLEARLY AND IN FULL SENTENCES. IMPROPER USE OF THE EQUAL SIGN WILL BE PENALIZED. NOT USING AN EQUAL SIGN WHERE IT SHOULD BE USED WILL ALSO BE PENALIZED. USE EXTRA PAPER AS NEEDED. IN OPTIMIZATION PROBLEMS, YOU MUST JUSTIFY THAT A MAXIMUM OR MINIMUM HAS BEEN ACHIEVED.**

The final exam will definitely be shorter, and probably easier, so don’t panic. I want to partially use these questions to review a lot of what we covered.

1. Compute the following limits. Use any procedure you want, but show ALL work.
   
   (a) \( \lim_{x \to 0} \frac{x - \tan x}{x^3} \).
   
   (b) \( \lim_{x \to 3} \frac{x^3 - 3x^2 + 2x - 6}{x^2 - 9} \).
   
   (c) \( \lim_{x \to \infty} \frac{(x^2 + 1)^3}{(x - 1)^2(2x + 1)^4} \).
   
   (d) \( \lim_{x \to 0} (1 + 3^x)^{1/x} \).

2. Evaluate the following derivatives. Since the assumption is that you did lot of exercises and now are at least as good as any calculator in finding derivatives, it is OK to just write the answer.

   (a) \( y = \sin \left( \frac{1}{x^2 + 1} \right) \), find \( \frac{dy}{dx} \).

   (b) \( f(x) = 2^x \sin x + \frac{x \sin x}{2x + 1} \), find \( f'(x) \).

   (c) \( g(x) = x \cos x \), find \( g'(x), g''(x), g'''(x) \).

   (d) \( g(x) = x \cos x \), find a general formula for the \( n \)-th derivative.

3. Find the tangent line to the curve of equation \( x^4 - 6xy^2 + y^4 = 25 \) at the point \( (3, 2) \).

4. For the following curves:

   (a) Determine ALL horizontal and vertical asymptotes.

   (b) Determine the intervals of increase and decrease.

   (c) Find all critical points and classify them as relative maximum, minimum, or neither.

   (d) Determine the intervals of concavity.

   (e) Find all inflection points.

   (f) Use the preceding information to sketch a graph of the curve. If the graph is inconsistent with your information you will get NO credit for the whole exercise.

   (a) \( y = x^3 - 12x^2 + 36x \).

   (b) \( y = \frac{\sin x}{2 + \cos x}, -\pi \leq x \leq \pi \).

   (c) \( y = \frac{x}{x^3 - 1} \).
5. (Exercise 4.7# 72 of the textbook) A rain gutter is to be constructed from a metal sheet of width 30 cm bending up one third of the sheet on each side through an angle $\theta$. How should $\theta$ be chosen so that the gutter will carry the maximum amount of water?

![Diagram of a rain gutter](image)

6. (Almost the same as 4.7, # 49) An oil refinery is located on the north bank of a straight river that flows east to west. The river is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 km east of the the refinery. The cost of laying pipe is $400,000/km over land to a point $P$ on the north bank and $750,000/km under the river to the tanks. To minimize the cost of the pipeline, where should $P$ be located?

7. (Exercise 4.7, # 16) A rectangular storage container with an open top is to have a volume of 10m$^3$. The length of the base is twice the width. Material for the base costs $10 per square meter. Material for the sides costs $6 per square meter. Find the cost of materials for the cheapest such container.

8. An object moves along a line. If its acceleration is given by $a(t) = \sin(\pi t)$ (in ft/s$^2$, $t$ being measured in seconds), and its initial velocity is 3 ft/s,

   (a) find its velocity function $v(t)$

   (b) If at time $t = 0$ it is at the origin ($s(0) = 0$), find its position at time $t = 2$ and the total distance traveled.

9. The height of a circular cone is increasing at the rate of 10 cm/s and the radius of its base at the rate of 5 cm/s. When the height is 80 cm and the radius of the base is 30 cm, how fast is the volume of the cone increasing. Recall that the formula for volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

10. Of the following formulas, 2 are true, 2 are false. Decide which are true, which false, and justify your answer!

   (a) $\int e^{x^2} \, dx = e^{x^2} + C$.

   (b) $\int \frac{1}{(1 + x^2)^2} \, dx = \frac{1}{2} \left( \frac{x}{1 + x^2} + \arctan x \right) + C$.

   (c) $\int \sin^3 x \, dx = \frac{1}{3} \cos^3 x - \cos x + C$.

   (d) $\int \frac{1}{1 + x^4} \, dx = \ln(1 + x^4) + C$.

11. Compute the following integrals.

   (a) $\int_0^1 x^2 \left( \sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 \, dx$.

   (b) $\int_0^3 \frac{2x^3}{x^4 + 5} \, dx$.

   (c) $\int \frac{e^x}{1 + e^{2x}} \, dx$. 
(d) $\int (x^4 + 1)^6 x^3 dx$.

12. Find the derivative of the following functions

(a) $g(x) = \int_{1}^{\sin x} \frac{1 - t^2}{1 + t^4} dt$

(b) $h(x) = \int_{\sqrt{x}}^{3x^2} \frac{e^t}{t} dt$

13. Show that the equation $2x - \cos x = 0$ has exactly one real root.