Calculus with Analytic Geometry I
Exam 8–October 21, 2011
Comments and Solutions

First of all, an important notice. Some of you have come up with a rule that is neither in the textbook, nor was mentioned in class: To find points in an interval where there is a maximum or a minimum, you check at the integers in the interval. The only reason this rule worked is that the author of our textbook has set up the problems so that all critical points are integers. So we have people finding NO critical point in an interval because they made a mistake computing a derivative, and yet miraculously finding a local maximum or minimum. Making an error in computing the derivative is forgivable; finding a local extremum and not recomputing the derivative having now full evidence of having made a mistake is not forgivable.

In future exams I will try to make sure that there always is at least one critical point that is not an integer. That may mean that the exercises in the test will not be exactly textbook exercises.

1. Find all the critical points of the following functions:

   (a) (15 points) \( f(x) = 4 + \frac{1}{2}x - \frac{1}{2}x^2. \)

      Solution. (4.1, # 29) \( f'(x) = \frac{1}{2} - x; \) setting it to 0 we get the single critical point \( x = \frac{1}{2}. \)

   (b) (15 points) \( f(x) = 2x^3 - 3x^2 - 36x. \)

      Solution. (4.1, # 31) \( f'(x) = 6x^2 - 6x - 36; \) setting to 0 and canceling a factor of 6 we get \( x^2 - x - 6 = 0; \) the roots are \(-3, 2\). The critical points are \( x = -3, 2.\)

   (c) (15 points) \( h(t) = t^{3/4} - 2t^{1/4}. \)

      Solution. This exercise (exercise 37 of Section 4.1) gave some people an inordinate amount of trouble. We have \( h'(t) = \frac{3}{4}t^{-1/4} - \frac{1}{2}t^{-3/4}. \) Setting to 0 and solving for \( t \) is not too bad, if one exercises a minimum of care. For example we could first get

      \[ \frac{3}{4}t^{-1/4} = \frac{1}{2}t^{-3/4}, \quad \text{then} \quad 3t^{-1/4} = 2t^{-3/4}. \]

      Now there are several things one can do. The easiest might be to multiply by \( t^{3/4}, \) use that \( t^{3/4}t^{-1/4} = t^{2/4} = \sqrt{t} \) to get \( 3\sqrt{t} = 2, \sqrt{t} = 2/3; \) squaring \( t = 4/9. \) Answering that the critical point was \( 4/9 \) gave you the full 15 points. Personally I think that is the answer, period. But our author opines differently. He gives as answer \( 0, 4/9. \) The problem with 0 is that this function is not defined for \( t < 0; \) powers like \( t^{1/4}, t^{3/4} \) are only defined for \( t \geq 0. \) The derivative does not make sense at 0. But one could say that if we restrict ourselves to the domain and talk of a derivative coming from the right at 0, then the function is not differentiable at 0. I also gave 15 points for answering \( t = 0, 4/9. \) For answering \( t = 0, \) I usually gave away 10 points.

2. Find the absolute maximum and the absolute minimum values of \( f \) in the given interval. Show where they occur and be sure to show all work.

   (a) (20 points) \( f(x) = 3x^4 - 4x^3 - 12x^2 + 1, \quad [-2, 3]. \)

      Solution. (4.1, # 51) \( f'(x) = 12x^3 - 12x^2 - 24x = 12x(x+1)(x-2). \) The critical points are \(-1, 0, 2.\)

      Testing at the critical points and at the endpoints one concludes that the maximum value is 33, assumed at \( x = -2, \) the minimum value is \(-31, \) at \( x = 2.\)

   (b) (20 points) \( f(x) = xe^{-x^2/8}, \quad [-1, 4]. \)

      Solution. (4.1, # 59) A lot of people had serious difficulties computing the derivative of \( f(x) = xe^{-x^2/8}. \) The derivative is

      \[ f'(x) = \frac{dx}{dx}e^{-x^2/8} = xe^{-x^2/8} \]

      \[ = 1 - e^{-x^2/8} + xe^{-x^2/8} \frac{d(-x^2/8)}{dx} = e^{-x^2/8} + xe^{-x^2/8} (-\frac{2x}{8}) = e^{-x^2/8} - \frac{x^2}{4} e^{-x^2/8}. \]
If you are still having difficulties with derivatives go slowly and, above all, DON’T IMPROVISE!

Setting the derivative to 0:

\[ 0 = e^{-x^2/8} - \frac{x^2}{4}e^{-x^2/8} = e^{-x^2/8} \left( 1 - \frac{x^2}{4} \right) \]

and since exponentials are never zero, we are left with \( 1 - \frac{x^2}{4} = 0 \), which solves to \( x = \pm 2 \). But -2 is not in the interval, so it can be discarded. The only critical point in the interval is 2. Now

\[
\begin{align*}
  f(-1) &= -e^{-1/8} \approx -0.883, \\
  f(2) &= 2e^{-1/2} \approx 1.213, \\
  f(4) &= 4e^{-2} \approx 0.541.
\end{align*}
\]

The absolute maximum is \( 2e^{-1/2} \), assumed at \( x = 2 \); the absolute minimum is \(-e^{-1/8}\) assumed at \( x = -1 \).

3. **(15 points)** Show that the equation \( 2x + \cos x = 0 \) has exactly one real root.

**Solution.** This is another exercise giving way more trouble than I was expecting. It is exercise 17 of Section 4.2 of the textbook and, since I did exercise 18 in full detail in class, I was expecting that most everybody would be able to do the similar # 17. Here is the solution I was looking for. It can’t be done without using words from the English language, strung together in the form of sentences. Part of my solution is in *italics*, that part is what I am thinking and I would not write this down if I were a student and was assigned this problem in an exam.

We should define a function by the left hand side of the equation. To see there is a zero I need to find points where the function changes sign. The easiest point to check is \( x = 0 \), there the function equals 1, so positive. To find a point where it is negative I might want to make \( x \) negative.

Let \( f(x) = 2x + \cos x \). We have \( f(0) = 1 > 0 \) and \( f(-\pi/2) = -\pi/2 < 0 \). Because \( f \) is continuous, there is \( c \) such that \(-\pi/2 < c < 0\) and \( f(c) = 0\). This shows the equation has at least one root.

To show there cannot be a second root I will have to use the Mean Value Theorem or better yet, Rolle’s Theorem. If there is a second root, then somewhere between the two roots, \( f' \) has to be zero.

The function \( f \) is differentiable and \( f'(x) = 2 - \sin x \). Since \( |\sin x| \leq 1 \) always, \( \sin x \) can never be 2 so \( f'(x) \) is never 0. If there were a second root \( d \) of the equation, then \( f(c) = f(d) = 0 \), and somewhere in between \( c \) and \( d \) there would have to be a point where \( f' \) is zero. So there cannot be a second root. And that is all!