Modern Analysis

Every Sequence has a Monotone Subsequence

Because my proof in class may have differed somewhat from the proof in the book, and because I may have fudged some details, here is a complete version.

**Theorem 1** Let \( \{a_n\} \) be a sequence of real numbers. Then \( \{a_n\} \) has a monotone subsequence.

**Proof.** **Case 1.** The sequence is not bounded above. In this case, we show it contains a strictly increasing subsequence diverging to \( \infty \). In fact, we can define the subsequence as follows. We let \( n_1 = 1 \). Assuming \( n_k \in \mathbb{N} \) defined for some \( k \geq 1 \), let

\[
A = \max\{a_1, \ldots, a_{n_k}, k\} = \max\{a_n : 1 \leq n \leq n_k\}, k).
\]

(So if all the terms \( a_1, \ldots, a_{n_k} \) are less than or equal \( k \), we will have \( A = k \). Otherwise it is the largest of the terms with index \( \leq n_k \).) Since the sequence is not bounded above, there exists an index \( m \in \mathbb{N} \) such that \( a_m > A \). We let \( n_{k+1} \) be such an index; that is \( n_{k+1} \) is such that \( a_{n_{k+1}} > A \). It should be clear that \( n_{k+1} > n_k \), because all the terms of the sequence with index up to and including \( n_k \) are \( \leq A \). This defines inductively the sequence of indices \( 1 = n_1 < n_2 < \ldots \); the definition shows that \( a_{n_{k+1}} > a_{n_k} \) for all \( k \) and \( a_{n_k} \geq k - 1 \) for \( k \geq 2 \). It follows that \( \{a_{n_k}\} \) is strictly increasing and \( \lim_{k \rightarrow \infty} a_{n_k} = \infty \).

**Case 2** The sequence is not bounded below. In this case there is a strictly decreasing subsequence diverging to \( -\infty \). The proof is, *mutatis mutandis*, the same as the proof for case 1 (what needs to be “mutated” are all the inequalities; they have to be reversed, and max replaced by min). Alternatively, it is an immediate consequence of applying Case 1 to the sequence \( \{-a_n\} \).

**Case 3.** Assume now the sequence is bounded. To prove that it has a monotone subsequence it suffices to prove that if it doesn’t have any increasing subsequence, then it must have a decreasing subsequence. So let us assume that \( \{a_n\} \) has no increasing subsequence.

**Claim:** For every \( N \in \mathbb{N} \), the set \( \{a_N, a_{N+1}, a_{N+2}, \ldots\} \) has a maximum element. Since I want this to be understood, I will rephrase the claim in two other equivalent ways. Claim: For every \( N \in \mathbb{N} \), the supremum of the set \( \{a_N, a_{N+1}, a_{N+2}, \ldots\} \) is an element of the set. Here’s another way. Claim: For every \( N \in \mathbb{N} \), there exists \( n \geq N \) such that \( a_m \leq a_n \) for all \( m \geq N \). And in case you don’t like these pesky quantifiers, the very last statement could be expressed by saying: if \( m \geq N \), then \( a_m \leq a_n \).

It is time to establish this claim. We proceed by contradiction saying in fact, assume the claim is false. There exists then \( N \in \mathbb{N} \) such that the set \( \{a_N, a_{N+1}, a_{N+2}, \ldots\} \) has no maximum element. But this is a bounded set (because the sequence is bounded), it is clearly not empty, thus it has a supremum. Let \( \beta = \sup\{a_N, a_{N+1}, a_{N+2}, \ldots\} \). Because the supremum is not in the set we have that \( a_n < \beta \) for all \( n \geq N \), a fact I’ll use several times. I will construct now an increasing subsequence of \( \{a_n\} \); a contradiction since we are assuming no such subsequence exists.

The idea is actually quite similar to that used in Case 1.

I begin taking \( n_1 = N \). Now \( a_N < \beta \). Every time a number is less than the sup of a set, there is something in the set larger than that number; thus there is \( n_2 \in \mathbb{N} \), \( n_2 \geq N \), \( a_{n_1} = a_N < a_{n_2} \). Obviously \( n_2 \neq N \), thus \( n_2 > N = n_1 \). We have \( n_1, n_2 \). As usual in defining subsequences in an abstract context, we proceed by induction. Assume \( n_k \) defined for some \( k \geq 1 \). Let

\[
A = \max\{a_N, a_{N+1}, \ldots, a_{n_2}\} = \max\{a_n : N \leq n \leq n_2\}.
\]

The difference with Case 1 is that we don’t add \( k \) to the list of objects of which \( A \) should be the maximum; \( A \) here is the largest term of the sequence with index between and including \( N \) and \( n_2 \). So \( A = a_j \) for some \( j \geq N \), hence \( A < \beta \). By the definition of \( \beta \), there is some index, we’ll call it \( n_{k+1} \geq N \) such that \( A < a_{n_{k+1}} \). For obvious reasons we must have \( n_{k+1} > n_k \) (at least, I hope the reasons are obvious) and \( a_{n_{k+1}} > a_{n_k} \). By induction, we have constructed a strictly increasing subsequence of \( \{a_n\} \), contradicting ur assumption that no such subsequence exists. The claim is established.

With this claim established, we can finally proceed to define our decreasing subsequence. The sub-indices for our decreasing sequence are defined as follows. We let \( n_1 \) be such that \( a_{n_1} = \sup\{a_n : n \in \mathbb{N}\} \). This makes sense.
thanks to our claim; the sup of the set \( \{a_n : n \in \mathbb{N}\} \) is, by the claim, an element of the set; i.e., a term of the sequence. There could be more than one index such that the term of that index is the sup; select (for example) the smallest such index. Or any other.

Assuming \( n_k \) defined for some \( k \geq 1 \), let \( n_{k+1} > n_k \) be such that

\[
a_{n_{k+1}} = \sup\{a_{n_{k+1}}, a_{n_{k+2}}, \ldots\} = \sup\{a_n : n \in \mathbb{N}, n > n_k\}.
\]

This makes sense by our claim. Having shown how to construct \( n_{k+1} \) from \( n_k \), the subsequence \( \{a_{n_k}\} \) can be considered as defined thanks to the magic of mathematical induction.

To conclude, we notice that \( \{a_{n_k}\} \) is decreasing; in fact,

\[
a_{n_k} = \sup\{a_n : n \in \mathbb{N}, n > n_{k-1}\}
\]

(also for \( k = 1 \) if we define \( n_0 = 0 \)) and since

\[
\{a_n : n \in \mathbb{N}, n > n_{k-1}\} \supseteq \{a_n : n \in \mathbb{N}, n > n_k\}
\]

we see that

\[
a_{n_k} = \sup\{a_n : n \in \mathbb{N}, n > n_{k-1}\} \geq \sup\{a_n : n \in \mathbb{N}, n > n_k\} = a_{n_{k+1}}
\]

for all \( k \in \mathbb{N} \). The proof is complete. \( \square \)