1. Not graded.

2. Here are some sequence exercises to play with. All in the friendly environment of the real numbers.

(a) Let \((a_n)\) be a sequence of non-negative real numbers. Consider the following set:

\[
S = \left\{ \sum_{n \in F} a_n : F \text{ is a finite subset of } \mathbb{N} \right\}.
\]

Just to be sure we are talking of the same thing, in case this definition was too abstract, \(S\) is the set of all numbers one obtains by adding a finite number of terms of the sequence. Thus \(S\) contains all terms of the sequence, all sums of two terms of the sequence \((a_n + a_m, n, m \in \mathbb{N}, n \neq m)\), all sums of three terms \((a_n + a_m + a_k, n, m, k \in \mathbb{N}, n \neq m \neq k \neq n)\), etc. Prove:

\[
\lim_{n \to \infty} \sum_{k=1}^{n} a_k = \sup S.
\]

**Hints/Suggestions:** We are assuming here implicitly that increasing sequences of real numbers either diverge to infinity or converge. You can let \(s = \lim_{n \to \infty} \sum_{k=1}^{n} a_k\); with \(s = \infty\) possible. One thing to notice is that \(\sum_{k=1}^{n} a_k \leq s\) for all \(n\). There are several approaches possible but one should remember that a standard way of proving \(a = b\) for real numbers \(a, b\) is by showing that the assumptions \(a > b\) and \(b > a\) lead to contradictions. Equivalently, by proving that \(a \leq b\) and then that \(b \leq a\).

**Solution.** Let \(s_n = \sum_{k=1}^{n} a_k\). We have to prove that \(\lim_{n \to \infty} s_n = \sup S\). The sequence \(\{s_n\}\) is increasing, thus \(s = \lim_{n \to \infty} s_n\) exists (possibly \(s = \infty\)); we need to prove \(s = \sup S\). Notice that \(s = \sup_{n \in \mathbb{N}} s_n\); in particular, \(s \geq s_n\) for all \(n\).

Let \(n \in \mathbb{N}\). Then, of course, \(F_n = \{1, \ldots, n\}\) is a finite set, thus \(\sum_{k \in F_n} a_k \in S\) and

\[
s_n = \sum_{k \in F_n} a_k \leq \sup S.
\]

Since \(s_n \leq \sup S\) for all \(n\), we also have \(s = \lim_{n \to \infty} s_n \leq \sup S\).

Assume now \(F\) is a finite subset of \(\mathbb{N}\). Let \(n = \max F\), then \(F_n \subset \mathbb{N}\),
{1, \ldots, n} \) and, since \( a_k \geq 0 \) for all \( k \),
\[
\sum_{k \in F} a_k \leq \sum_{k=1}^{n} a_k \leq s_n \leq s.
\]
Since \( F \) was an arbitrary finite subset of \( \mathbb{N} \), this shows that \( s \) is an upper bound of \( S \), hence \( \sup S \leq s \). We proved \( s \leq \sup S \) and \( \sup S \leq s \), hence we are done.

(b) Given a **bounded** sequence \((a_n)\) of real numbers, consider the set \( L \) defined by
\[
L = \{ \ell \in \mathbb{R} : \text{there exists a subsequence of} \ (a_n) \ \text{converging to} \ \ell \}.
\]
By the Bolzano Weierstrass theorem, \( L \neq \emptyset \). To fix ideas, here are some examples. The exercise as such begins after the examples. The examples are here to make sure we understand what is meant by \( L \).

**Example 1.** Let \((a_n)\) be a sequence converging to \( a \in \mathbb{R} \). Then \( L = \{a\} \).

**Example 2.** Let \( a_n = \frac{(-1)^{n+1}n}{n+1} \). Then \( L = \{-1, 1\} \).

**Example 3.** Let \( a_n = \sin(n\pi/4) \). Then \( L = \{-1, -1/\sqrt{2}, 0, 1/\sqrt{2}, 1\} \).

Here is the exercise.

i. Prove: The set \( L \) is a closed subset of \( \mathbb{R} \).

**Solution.** Let \( x \in L' \) (a cluster point of \( L \)). We will construct a subsequence of \((a_n)\) converging to \( x \) as follows.

**Possible Pitfall** One thing that needs to be done in getting a subsequence is to make sure one has a strictly increasing sequence of indices. For example, let us assume that the sequence is \( a_n = (-1)^{n+1} \) so that \( L = \{1, -1\} \). If we want to find a subsequence converging (say) to \( 1 \in \mathbb{L} \), one has to avoid taking \( n_k = 1 \) for all \( k \), because that is not a subsequence. One has to make sure of two things: 1) The final product \((a_{n_k})\) is a subsequence of \((a_n)\). 2) It converges to \( x \).

My solution may be a bit too verbose, but it is very hard (I think) to avoid some type of induction argument just to make sure one has a subsequence. **To repeat:** It is not enough to get \( a_{n_k} \) close to \( x \); one also needs \( n_k > n_{k-1} \).

By definition of cluster point there is \( x_1 \in L \) such that \( |x_1 - x| < 1 \). Since \( x_1 \in L \) it is the limit of some subsequence \((a_{n_k})\) of \( a_n \); there is then \( K \) such that \( k \geq K \) implies \( |a_{n_k} - x_1| < 1 - |x_1 - x| \) so that \( |a_{n_k} - x| < 1 \). Fix \( n_k \) for some \( k \geq n_k \) and call it \( m_1 \); so \( m_1 \in \mathbb{N} \) and \( |a_{m_1} - x| < 1 \).

Assume now selected integers \( m_1, \ldots, m_\ell \) for some \( \ell \geq 1 \) such that \( 1 \leq m_1 < m_2 < \cdots < m_\ell \) and such that \( |a_{m_j} - x| < 1/j \) for \( j = 1, \ldots, \ell \). (This has been done for \( \ell = 1 \).) Recalling that \( x \) is a
Define a sequence \((a_n)\) such that \(|a_n - x_0| < 1/(\ell + 1)\). There is a subsequence, we may denote it again by \((a_{n_k})\), of \((a_n)\) converging to \(x_0\); there is thus \(K\) such that if \(k \geq K\) then \(|a_{n_k} - x_0| < 1/(\ell + 1)\). Select \(k > \max(K, m_\ell)\); set \(m_{\ell+1} = n_k\). Then \(m_{\ell+1} = n_k \geq k > m\ell\) and \(|a_{m_{\ell+1}} - x| < 1/(\ell + 1)\). In this way we construct inductively a subsequence \((a_{m_n})\) of \((a_n)\) satisfying \(|a_{m_n} - x| < 1/j\), thus converging to \(x\). It follows that \(x \in L\).

Incidentally, the fact that the sequence was bounded played no role. I only added it to make everybody feel more comfortable. It assures that \(L \neq \emptyset\); that is all. The result is actually valid in all metric spaces, not only in \(\mathbb{R}\).

ii. Define a sequence \((a_n)\) as follows:

\[
a_n = \begin{cases} 
1, & \text{if } n \text{ is not divisible by 3,} \\
\min(\frac{1}{2}, 10), & \text{if } n = 2^r3^m, \gcd(m, 6) = 1, e, f \in \mathbb{Z}, \quad e \geq 0, \quad f > 0.
\end{cases}
\]

A. Prove that every non-negative rational number \(\leq 10\) appears an infinite number of times in the sequence.

B. Prove that \(L = [0, 10]\)

(Here 10 was an arbitrary cutoff to keep the sequence bounded).

**Solution.** Concerning the first part, if \(x \in [0, 10] \cap \mathbb{Q}\) we can write \(x = m/n\) where \(m \in \mathbb{Z}\), \(m \geq 0\), \(n \in \mathbb{N}\). For \(k \in \mathbb{N}\) let \(n_k = 2^k3^m\), then \(a_{nk} = mk/nk = m/n = x\) for \(k = 1, 2, 3, \ldots\), so \(x\) appears an infinite number of times in the sequence and, for future reference, the subsequence \((a_{nk})\) converges to \(x\). This proves also that \(\mathbb{Q} \cap [0, 10] \subset L\). Since \(L\) is closed by a previous exercise (and \(L \subset [0, 10]\), it follows that \(L = [0, 10]\).

3. Let \(M\) be a metric space and let \(A, B\) be compact subsets. Prove: \(A \cup B\) is compact.

**Solution.** Let \(U\) be an open covering of \(A \cup B\). Let \(U_1 = \{U \in U : A \cap U \neq \emptyset\}\) and let \(U_2 = \{U \in U : B \cap U \neq \emptyset\}\). Clearly \(U_1\) is an open covering of \(A\) and \(U_2\) is an open covering of \(B\). Since \(A, B\) are compact, we can find \(U_1, \ldots, U_m \in U_1\) such that \(A \subset \bigcup_{k=1}^m U_k\), and \(U_{m+1}, \ldots, U_{m+r} \in U_2\) such that \(B \subset \bigcup_{k=m+1}^{m+r} U_k\). Then \(\{U_1, \ldots, U_{m+r}\}\) is a finite subcovering of \(U\) covering \(A \cup B\).

4. This is essentially Exercise 36 of Chapter 2 in Pugh’s book. It requires a definition. Let \(S\) be a set and \(f : S \to S\). If \(a \in S\), then the orbit of \(a\) is the subset \(\text{orb}(a) = \{a, f(a), f(f(a)), f(f(f(a))), \ldots\}\). For example, if \(S = \mathbb{R}\) and \(f(x) = x^2\), then \(\text{orb}(1) = \{1\}, \text{orb}(2) = \{2, 4, 16, 256, 65536, \ldots\}\).

Let \(\alpha \in [0, 2\pi]\) and let \(R\) be the rotation of the unit circle by an angle \(\alpha\). (In polar coordinates, the angle coordinate \(\theta\) gets sent to \(\theta + \alpha\).) Prove:
If $\alpha/\pi \in \mathbb{Q}$, then all orbits are finite sets. If $\alpha/\pi \not\in \mathbb{Q}$, then all orbits are infinite sets, dense in the circle.

**Suggestions** The first part, the part that appears unstarrred in Pugh, that orbits are finite if $\alpha/\pi$ is rational, is quite easy. It means there exist positive integers $m, n$ such that $m\alpha = 2n\pi$ and it should be easy to prove that each orbit has lcm$(m, n)$ elements. For the second part it is also easy, I think, to show that all orbits are (countably) infinite. The hard part is to see that the closure of each orbit is the circle. You may (and probably should) use that the map $\Phi: \mathbb{R} \to \mathbb{R}^2$ defined by $\Phi(t) = (\cos t, \sin t)$ for $t \in \mathbb{R}$, maps the real line continuously onto the unit circle. Looking at the inverse image; i.e., pre-image, of an orbit under this map could help relate this problem to a problem from a previous homework.

**Solution.** Assume first $\alpha/\pi \in \mathbb{Q}$, say $\alpha = m/n$ where $m, n \in \mathbb{N}$. (The case $m = 0$ corresponds to $\alpha = 0$, hence $R = \text{id}$, so all orbits are singleton sets. We may assume $m > 0$.) I think complex notation can be quite convenient here; points of unit circle being of the form $e^{i\theta}$, $\theta \in \mathbb{R}$. So if $z = e^{i\theta}$ is in the unit circle, then writing $R^k z$ for $R$ composed $k$ times with itself, we have $R^k z = e^{i(k+j)}$ from which we see that

$$R^{2n} z = e^{i(\theta+2m\pi)} = e^{i\theta} = z.$$  

It follows that $R^{2n}$ is the identity map (if $n$ is even, then $R^n$ is already the identity), thus every orbit contains at most $2n$ elements.

Assume now $\alpha/\pi \not\in \mathbb{Q}$. The fact that each orbit is infinite is not too hard. Consider the orbit containing $z = e^{i\theta}$. For it to be finite we must have $R^k z = R^j z$ for some integers $0 \leq k \neq j$. We may assume $k > j$, and see that this works out to $e^{i(k-j)\alpha} = 1$, hence $(k-j)\alpha = 2m\pi$ for some $m \in \mathbb{N}$, hence $\alpha/\pi = 2m/(k-j) \in \mathbb{Q}$, a contradiction.

For the rest, let $\beta = \alpha/(2\pi)$. By Exercise 1(d) of Homework 1, we get that $G = \{m+n\beta : m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Well, to be quite fair, Exercise 1(d) does not quite go so far; however, I mentioned this fact in class as a consequence of this exercise. The set $G$ clearly is a non trivial additive subgroup of $\mathbb{R}$ and the alternative to being dense is that it is of the form $\gamma\mathbb{Z}$ for some $\gamma \in \mathbb{R}$. Since $1 = 1 + 0\beta, \beta = 0 + 1\cdot \beta \in G$, this would imply the existence of integers $k, \ell$ such that $1 = k\gamma$ and $\beta = \ell\gamma$; then $\beta = \beta/1 = \ell/k \in \mathbb{Q}$, a contradiction. So having seen that $G$ is dense, let $z = e^{i\theta} = (\cos \theta, \sin \theta) \in C$; where $C$ is the unit circle. I want to prove the orbit of $z$ is dense in $G$. So let $w = (\cos \varphi, \sin \varphi)$ be any (other) point of $C$; we may assume $0 \leq \theta, \varphi < 2\pi$. Here it gets a little complicated.

Because $G$ is dense in $\mathbb{R}$, so is the set $\tilde{G} = \{m/n : m, n \in \mathbb{Z}\}$; it is just $G$ shifted by $\theta/(2\pi)$. There are thus subsequences $(m_k), (n_k)$ of integers such that $\lim_{k \to \infty} (\frac{\theta}{2\pi} + m_k + n_k\beta) = \varphi/(2\pi)$. The map $x \mapsto$
(\cos(2\pi x), \sin(2\pi x)) : \mathbb{R} \rightarrow C \text{ is continuous thus}

\lim_{k \to \infty} \left( \cos 2\pi \left( \frac{\theta}{2\pi} + m_k + n_k \beta \right), \sin 2\pi \left( \frac{\theta}{2\pi} + m_k + n_k \beta \right) \right) = (\cos \varphi, \sin \varphi) = w.

But

\left( \cos 2\pi \left( \frac{\theta}{2\pi} + m_k + n_k \beta \right), \sin 2\pi \left( \frac{\theta}{2\pi} + m_k + n_k \beta \right) \right)

= (\cos (\theta + 2\pi m_k + n_k \alpha), \sin (\theta + 2\pi m_k + n_k \alpha)) - (\cos (\theta + n_k \alpha), \sin (\theta + n_k \alpha)) = R^n z

is in the orbit of \( z \). The proof is complete.

5. The first problem from the section of Pre-lim problems in Pugh is quite interesting. Suppose \( f : \mathbb{R}^m \rightarrow \mathbb{R} \) satisfies two conditions:

(a) For each compact subset \( K \) of \( \mathbb{R}^m \), \( f(K) \) is compact.

(b) For any nested decreasing sequence of compact sets \( (K_n) \),

\[
f \left( \bigcap_{n=1}^{\infty} K_n \right) = \bigcap_{n=1}^{\infty} f(K_n).
\]

Prove that \( f \) is continuous.

Solution. A key ingredient in this proof is a result on compactness not seen in class. It would have been given as a hint, or developed in the problem solving session. It is:

**Lemma.** Let \( M \) be a metric space, let \( C_n \) be a compact subset of \( M \) for \( n \in \mathbb{N} \), let \( U \) be open and assume that \( \bigcap_{n=1}^{\infty} C_n \subset U \). Then there exists \( N \in \mathbb{N} \) such that \( \bigcap_{n=1}^{N} C_n \subset U \). (If an open set contains the intersection of a family of compact sets, it already contains the intersection of a finite subfamily).

**Proof.** Let \( K_n = \left( \bigcap_{k=1}^{n} C_k \right) \setminus U \). Intersection of compact sets being compact, complement of open sets being closed, and closed subsets of a compact set being compact, we see that \( K_n \) is compact for each \( n \in \mathbb{N} \). It is also clear that \( K_1 \supset K_2 \supset K_3 \supset \cdots \) and that

\[
\bigcap_{n=1}^{\infty} K_n = \left( \bigcap_{n=1}^{\infty} C_n \right) \setminus U = \emptyset.
\]

By the “decreasing sequence of nested compact sets” theorem, there is \( N \) such that \( K_N = \emptyset \), which works out to \( \bigcap_{n=1}^{N} C_n \subset U \).

We now have several ways of attacking this exercise. I am giving two solutions.
Solution I. By sequences. We show: If $x \in \mathbb{R}^m$ and $(x_n)$ is a sequence converging to $x$, then $(f(x_n))$ converges to $x$. Assume thus that $x \in \mathbb{R}^m$ and $(x_n)$ is a sequence converging to $x$. It is then easy to see (using Heine Borel, for example, which is valid in $\mathbb{R}^m$) that the sets $K_n = \{x_k : k \geq n\} \cup \{x\}$ form a decreasing sequence of compact sets. Thus all the sets $f(K_n)$ are compact by the first hypothesis while

$$\bigcap_{n=1}^{\infty} f(K_n) = f\left(\bigcap_{n=1}^{\infty} K_n\right) = f(\{x\}) = \{f(x)\}.$$  

(The fact that $\bigcap_{n=1}^{\infty} K_n = \{x\}$ does require a bit of a proof. It is easy to see that the intersection must contain $x$; anything else in the intersection would have to be an element $y$ which is equal to $x_n$ for an infinite number of $n$'s; but since the sequence converges to $x$, that is not possible if $y \neq x$.)

To see that sequence $\{f(x_n)\}$ converges to $f(x)$, let $\epsilon > 0$ be given. Then

$$\bigcap_{n=1}^{\infty} f(K_n) = \{f(x)\} \subset (f(x) - \epsilon, f(x) + \epsilon),$$

thts, by the Lemma, there is $N$ such that $f(K_N) = \bigcap_{n=1}^{N} f(K_n) \subset B(f(x), \epsilon)$, which implies $|f(x_k) - f(x)| < \epsilon$ if $k \geq N$.

Solution II. Let $x \in \mathbb{R}^m$. For $n \in \mathbb{N}$, the sets $B(p, 1/n) = \{y \in \mathbb{R}^m : |y - x| \leq 1/n\}$ are closed and bounded, hence compact. Moreover they form a decreasing sequence (of compact sets) with intersection equal to $x$. By hypothesis

$$\bigcap_{n=1}^{\infty} f(B(x, 1/n)) = \{f(x)\}.$$  

Assume now $V$ is a neighborhood of $f(x)$ in $\mathbb{R}$; then it contains an open subset of $\mathbb{R}$ that contains $f(x)$; by hypothesis all the sets $f(B(x, 1/n))$ are compact thus, by the Lemma, there is $N$ such that $f(B(x, 1/N)) = \bigcap_{n=1}^{N} f(B(x, 1/n)) \subset V$. Since $B(x, 1/n)$ is a neighborhood of $x$ in $\mathbb{R}^m$ we proved that for each neighborhood $V$ of $f(x)$ in $\mathbb{R}$ there is a neighborhood of $x$ in $\mathbb{R}^m$ mapped by $f$ into $V$, hence $f$ is continuous at $x$. Since $x$ was arbitrary in $\mathbb{R}^m$ we are done.