1. In $\mathbb{R}^m$ consider the following three distance functions, defined by

$$
\begin{align*}
d_1(x, y) &= \sum_{i=1}^{m} |x_i - y_i| \\
d_2(x, y) &= \left( \sum_{i=1}^{m} |x_i - y_i|^2 \right)^{1/2} \\
d_{\infty}(x, y) &= \max_{1 \leq i \leq m} |x_i - y_i|.
\end{align*}
$$

for $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in \mathbb{R}^m$.

(a) Assume $(x_n)$ is a sequence in $\mathbb{R}^m$ where $x_n = (x_{n1}, \ldots, x_{nm})$ for $n = 1, 2, 3, \ldots$. **Prove** the sequence $(x_n)$ converges with respect to $d_1$ to $x \in \mathbb{R}^m$ if and only if it converges with respect to $d_2$ to $x$, and this happens if and only if it converges with respect to $d_{\infty}$ to $x$. This is an $\epsilon$ manipulation exercise.

(b) Assume $(x_n)$ is a sequence in $\mathbb{R}^m$ where $x_n = (x_{n1}, \ldots, x_{nm})$ for $n = 1, 2, 3, \ldots$ and let $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$. **Prove** the sequence $(x_n)$ converges to $x$ with respect to $d_2$ if and only if each one of the $m$ sequences $(x_{nk})_{n \in \mathbb{N}}$ converges to $x_k$ in $\mathbb{R}$, $k = 1, \ldots, m$.

**Note:** Except if otherwise mentioned, we ALWAYS consider $\mathbb{R}$ as a metric space with the metric $d(x, y) = |x - y|$. Except if otherwise mentioned, we ALWAYS consider $\mathbb{R}^m$ as a metric space with the metric $d = d_2$. But part (a) of this exercise is basically showing that one might use $d_1$ or $d_{\infty}$ instead. The three distance functions $d_1, d_2,$ and $d_{\infty}$ are said to be equivalent. Part (b) shows that one can reduce a lot, maybe most, of the metric considerations in $\mathbb{R}^m$ to considerations in $\mathbb{R}$.

2. Let $M$ be a metric space with distance function $d$. Let $(p_n)$ be a sequence in $M$.

(a) **Prove** that a subsequence of a subsequence of $(p_n)$ is a subsequence of $(p_n)$.

(b) **Prove** that $\lim_{n \to \infty} p_n = p$ if and only if $\lim_{n \to \infty} p_{2n} = p$ and $\lim_{n \to \infty} p_{2n+1} = p$.

3. (a) Let $M$ be a set and let $d : M \times M \to [0, \infty)$ satisfy for $p, q, r \in M$

i. $d(p, q) \geq 0$ and $d(p, q) = 0$ if and only if $p = q$.

ii. $d(p, q) = d(q, p)$.
iii. \( d(p, r) \leq \max(d(p, q), d(q, r)) \).

**Prove** \( d \) is a metric (distance function) for \( M \). Such a metric is called an **ultrametric**.

(b)* Let \( p \) be a prime number. If \( x \in \mathbb{Q} \) define \( |x|_p \) as follows. If \( x \neq 0 \) write

\[
x = p^{\frac{k_m}{n}}
\]

where \( m, n, k \in \mathbb{Z}, n \neq 0, \gcd(m, p) = \gcd(n, p) = 1 \). The number \( k \) is uniquely determined; define \( |x|_p = p^{-k} \). For example, if \( p = 3 \) we have:

\[
|9|_3 = 3^{-2} = 1/9, \quad |5/6|_3 = 3, \quad |1/2|_3 = 1, \quad |7|_3 = 1, \quad |36/5| = 1/9, \quad \text{etc}
\]

If \( x = 0 \), set \( |0|_p = 0 \). If \( x, y \in \mathbb{Q} \) define \( d_p(x, y) = |x - y|_p \). **Prove** \( d_p \) is an ultrametric for \( \mathbb{Q} \). It is called the **\( p \)-adic** metric.

4. Let’s go back to the notion of equivalence for metrics. Two distance functions \( d_1, d_2 \) for a set \( M \) are said to be equivalent iff a sequence converges with respect to \( d_1 \) to a point \( p \in M \) if and only if it converges with respect to \( d_2 \) to \( p \). A metric space \((M, d)\) is said to be **discrete** iff its metric is equivalent to the discrete metric. **Prove** \( \mathbb{N} \) as a metric subspace of \( \mathbb{R} \) is discrete.

5. Let \( M \) be a set and suppose \( d_1, d_2 \) are distance functions in \( M \). **Prove** they are equivalent iff the identity function from \( M \) to \( M \) is a homeomorphism from the metric space \((M, d_1)\) to the metric space \((M, d_2)\).

