1. Because of the completeness axiom, given any set of real numbers that is not empty and bounded above, we can immediately talk of the supremum of the set. Similarly, if it is not empty and bounded below, we can immediately talk of the infimum. Thus, given a set $S \subseteq \mathbb{R}$, if we have proved somehow that there is $x \in S$ (so $S \neq \emptyset$) and there is $b \in \mathbb{R}$ such that $x \leq b \ \forall x \in S$, then it becomes legal to say, for example, Let $s = \sup S$.

The following exercises have you practice with suprema and infima.

(a) Let $S \subseteq \mathbb{R}$, assume $S \neq \emptyset$ and $S$ is bounded above. Let $\sigma = \sup S$. 
Prove: For each $\epsilon \in \mathbb{R}$, $\epsilon > 0$ show there exists $x \in S$ such that $\sigma - \epsilon < x \leq \sigma$. Can one always improve this to: There exists $x \in S$ such that $\sigma - \epsilon < x < \sigma$? (This is exercise 12, parts (a), (b), of Chapter 1 of our textbook.)

Solution. Let $\epsilon > 0$ be given. Then $\sigma - \epsilon < \sigma$, thus $\sigma - \epsilon$ cannot be an upper bound of $S$, hence there is $x \in S$ such that $x > \sigma - \epsilon$. Since $\sigma$ is an upper bound of $S$, we must have $x \leq \sigma$. This proves the existence of $x \in S$, $\sigma - \epsilon < x \leq \sigma$. The last inequality cannot be “improved” to be strict; for example if $S$ contains a single element, say $S = \{1\}$, then $\sigma = x = 1$.

(b) The set $\mathbb{Q}$ is an ordered field, clearly $\mathbb{Q}$ is not bounded. However, if we think of $\mathbb{Q}$ as a subfield of some larger ordered field, there is no reason why $\mathbb{Q}$ should not have some upper bound or lower bound in that larger field. Thus, it is not a given (at first) that, as a subset of $\mathbb{R}$, $\mathbb{Q}$ is bounded. Well, it obviously is, but maybe it isn’t so obvious.

The question here is: Does the fact that $\mathbb{Q}$ is not bounded as a subset of $\mathbb{R}$ follow from the axioms? What should be clear is that it all reduces to the question of whether $\mathbb{N}$ is bounded above. If it isn’t, neither is the larger set $\mathbb{Q}$. And $-\mathbb{N}$ would be unbounded below, so would $\mathbb{Q}$. Prove that $\mathbb{N}$ is NOT bounded above.

Hint: Assume it is, and use the completeness axiom.

Solution. For a contradiction, assume $\mathbb{N}$ is bounded above. Since $\mathbb{N}$ obviously not empty, by the completeness axiom $\sigma = \sup \mathbb{N}$ exists. Then $\sigma - 1 < \sigma$, thus there exists $n \in \mathbb{N}$ such that $\sigma - 1 < n$ so that $\sigma < n + 1$. But $n + 1 \in \mathbb{N}$, thus $n + 1 > \sigma$ contradicts that $\sigma$ is an upper bound of $\mathbb{N}$.

A slightly different proof is as follows, once one has brought $\sigma = \sup \mathbb{N}$ into the picture: Let $n \in \mathbb{N}$. Then $n + 1 \in \mathbb{N}$, hence $n + 1 \leq \sigma$, proving
that \( n + 1 \leq \sigma \) for all \( n \in \mathbb{N} \). But then \( n \leq \sigma - 1 \) for all \( n \in \mathbb{N} \), hence \( \sigma - 1 \) is an upper bound of \( \mathbb{N} \), contradicting that \( \sigma \) is the least upper bound of \( \mathbb{N} \).

(c) Obtain as a consequence of the fact that \( \mathbb{N} \) is not bounded above the following result: Let \( \epsilon \in \mathbb{R} \) and assume \( \epsilon > 0 \). There exists \( n \in \mathbb{N} \) such that \( 1/n < \epsilon \).

**Solution.** Let \( \epsilon > 0 \) be given. Since \( \mathbb{N} \) is not bounded above, there is \( n \in \mathbb{N} \) such that \( n > 1/\epsilon \). This implies \( 1/n < \epsilon \).

(d) A non-empty subset \( G \) of \( \mathbb{R} \) is an additive subgroup of \( \mathbb{R} \) if and only if it is closed under addition and with addition it is a group. In symbols:

\[
G \subset \mathbb{R} \text{ is an additive subgroup iff } 1) \ 0 \in G; \ 2) \ x, y \in G \Rightarrow x + y \in G, \ 3) \ x \in G \Rightarrow -x \in G.
\]

There is a shorter equivalent way: iff \( G \neq \emptyset \) and \( x, y \in G \Rightarrow x - y \in G \). In fact, If \( G \neq \emptyset \), then there is \( x \in G \), hence \( x, x \in G \), hence \( x - x = 0 \in G \). If now \( x \in G \), then \( 0, x \in G \), hence \( -x = 0 - x \in G \). Finally, \( x, y \in G \) implies \( x, -y \in G \), hence \( x + y = x - (-y) \in G \).

Examples of additive subgroups are: \( \{0\} \), \( \mathbb{Z} \), \( \mathbb{Q} \), and \( \mathbb{R} \) itself.

In this exercise you are asked to classify all additive subgroups of \( \mathbb{R} \) by following instructions.

i. Let \( G \) be an additive subgroup of \( \mathbb{R} \), \( G \neq \{0\} \). Define \( G^+ = \{x \in G : x > 0\} \). Prove \( G^+ \neq \emptyset \) and is bounded below.

**Solution.** Since \( G \neq \{0\} \), there is \( x \in G, x \neq 0 \). Then \( -x \in G \) and one of \( x, -x \) must be positive, thus \( G^+ \neq \emptyset \). That \( G^+ \) is bounded below by \( 0 \) is obvious, given the definition of \( G^+ \).

ii. Let \( \alpha = \inf G^+ \). Prove: If \( \alpha > 0 \), then \( \alpha \in G^+ \) and \( G = \alpha \mathbb{Z} = \{k\alpha : k \in \mathbb{Z}\} \).

**Hint:** There has to be a \( G^+ \) element \( x \) very close to \( \alpha \). If \( x > \alpha \), there has to be another \( G^+ \) element even closer; \( \alpha < y < x \). Now \( x - y \in G \) and one obtains a contradiction if one took \( x \) close enough to \( \alpha \).

**Solution.** Let \( \alpha \inf G^+ \) and assume \( \alpha > 0 \). Then \( 2\alpha > \alpha \), hence \( 2\alpha \) is not a lower bound of \( G^+ \) and there exists \( x \in G^+ \) such that \( \alpha \leq x < 2\alpha \). If \( \alpha \notin G^+ \), then \( x > \alpha \) and there exists \( y \in G^+ \) such that \( x > y > \alpha \). But \( x, y \in G^+ \) implies \( x - y \in G \), hence \( x - y \in G^+ \) since \( x > y \). But we also have \( \alpha < y < x < 2\alpha \) and this implies \( x - y < \alpha \). It follows that \( \alpha \in G^+ \). Since \( G \) is a group and \( \alpha \in G \), we have \( \alpha \mathbb{Z} \subset G \). For the converse inclusion, let \( x \in G \). Since \( \alpha > 0 \), there exists \( k \in \mathbb{Z} \) such that \( k\alpha \leq x < (k + 1)\alpha \). This fact could require a proof in itself; I append a proof below. Then \( 0 \leq x - k\alpha < \alpha \). It follows that \( x = k\alpha \); otherwise we would have \( 0 < x - k\alpha < \alpha \), contradicting \( \alpha = \inf G^+ \), since \( x - k\alpha \in G \).
iii. Assume now \( \alpha = 0 \). Show that for each \( \epsilon \in \mathbb{R}, \epsilon > 0 \), there is \( x \in G \) such that \( 0 < x < \epsilon \).

**Solution.** Trivial! But it sets the step for the next result.

iv. Assume \( \alpha = 0 \). Show that if \( x, y \in \mathbb{R}, x < y \), there exists \( z \in G, x < z < y \).

**Solution.** Assume \( \alpha = 0 \) and \( x, y \in \mathbb{R}, x < y \). By the previous result (the trivial one), since \( y - x > 0 \), there exists \( g \in G \) such that \( 0 < g < y - x \). Let \( m \in \mathbb{Z} \) be such that \( mg \leq x < (m + 1)g \) (for the existence of such an \( m \), see the proof following this exercise). Because \( G \) is a group, \( (m + 1)g \in G \); set \( z = (m + 1)g \). We already have \( x < z \). Now, since \( mg \leq x \),

\[
z = mg + g \leq x + g < x + (y - x) = y.
\]

The conclusion is that every non-trivial additive subgroup \( G \) of \( \mathbb{R} \) is either dense (there is a group element between every pair of distinct real numbers) or consists of all the integer multiples of a positive number.

**Proof of a result used in parts ii and iv of part (d) of Exercise 1:** namely,

Let \( \beta \in \mathbb{R}, \beta > 0 \). For each \( x \in \mathbb{R} \), there exists a unique \( n \in \mathbb{Z} \) such that \( n \beta \leq x < (n + 1) \beta \).

**Proof.** Let \( x \in \mathbb{R} \). One approach (of many) is as follows. Consider the set \( S = \{ k \in \mathbb{Z} : k \leq x/\beta \} \). This set is not empty (since \( \mathbb{N} \) is not bounded above, it follows easily that \( \mathbb{Z} \) is not bounded below; there is \( k \in \mathbb{Z}, k \leq x/\beta \); it is obviously bounded above (by \( x/\beta \)). Let \( n = \sup S \).

I claim \( n \in \mathbb{Z} \). In fact, assume \( n \notin \mathbb{Z} \). Because \( n = \sup S \), there is \( k \in S \) such that \( n - 1 < k \leq n \). If \( j \in \mathbb{Z} \) and \( j > k \), then \( j \geq k + 1 > n \) and \( j \notin S \). But \( S \) is a set of integers, so if \( j \in \mathbb{Z}, j > k \) implies \( j \notin S \), it follows that \( k \) is an upper bound of \( S \), hence \( k \geq n \). It follows that \( k = n \). Notice the similarity with the proof of the Archimedean property, part (b) of exercise 1. In fact, we proved: If \( S \) is a set of integers that is bounded above, then \( \sup S = \max S \in S \).

Since \( n \in S \), we see that \( n \in \mathbb{Z} \) and \( n \leq x/\beta \), thus \( n \beta \leq x \) since \( \beta > 0 \).

Now that we have that \( n \in \mathbb{Z} \), we must also have \( n + 1 > x/\beta \); otherwise \( n + 1 \in S \) contradicting \( n = \sup S \). Thus \( (n + 1)\beta > x \). \[\Box\]

2. **Textbook, Exercise 19, Chapter 1, Page 43.**

**Solution.**

(a) One has to show that a function \( f : A \to A \) has a fixed point if and only if \( G_f = \{(a, f(a)) : a \in A\} \), the graph of \( f \), intersects the diagonal \( D = \{(a, a) : a \in A\} \). That is, of course, quite immediate: If \( f \) has a fixed point, there is \( a \in A \) with \( f(a) = a \). Now \((a, f(a)) \in\)
Assume now $G_J f$ but, because $f(a) = a$, we also have $(a, f(a)) = (a, a) \in D$. Thus $G_J f \cap D \neq \emptyset$. Conversely, assume a $G_J f \cap D \neq \emptyset$ and let $(a, b) \in G_J f \cap D$. Since $(a, b) \in G_J$, $b = f(a)$. Since $(a, b) \in D$, $b = a$. Thus $f(a) = b = a$.

(b) Assume now $f : [0, 1] \rightarrow [0, 1]$ is continuous. Since $f$ assume values in $[0, 1]$, we have $f(0) \geq 0$ and $f(1) \leq 1$. Let $g : [0, 1] \rightarrow \mathbb{R}$ be defined by $g(x) = f(x) - x$. Because $f$ is continuous, so is $g$. Moreover, $g(0) = f(0) - 0 \geq 0$; $g(1) = f(1) - 1 \leq 0$; by the intermediate value theorem there is $c \in [0, 1]$ such that $g(c) = 0$; i.e., $f(c) = c$.

(c) No; if $f : (0, 1) \rightarrow (0, 1)$ is continuous, it does not necessarily have a fixed point. An obvious counterexample is $f(x) = x^2$.

(d) No; if $f : [0, 1] \rightarrow [0, 1]$ is not continuous, it can go from 0 to 1 without crossing the diagonal, by just jumping. An obvious counterexample is

$$f(x) = \begin{cases} 
1, & 0 \leq x < 1/2, \\
0, & 1/2 \leq x \leq 1.
\end{cases}$$

3. Textbook, Exercise 22, Parts (a), (b), Chapter 1, Page 44.

**Solution of part a.** Let $\epsilon > 0$ be given. We may, of course, assume $\epsilon < \pi$, otherwise the problem reduces to full blown triviality. Let $r \in \mathbb{R}$ be such that $\sqrt{1 - \frac{\pi}{2}} < r < 1$. It may be useful to introduce the notation $D$ for the closed unit disc and $D_r$ for the closed disc of radius $r$.

I will begin claiming first a fact that is fairly obvious geometrically (i.e., by pictures): If $Q$ is a square (dyadic or not) of sides parallel to the axes, each side of length $\delta$, if $0 < \delta < (1-r)^2/4$, and if $Q \cap D_r \neq \emptyset$, then $Q \subset D$. In fact, assume all of this, and let $(x_0, y_0) \in Q \cap D_r$. We want to prove: If $(x, y) \in Q$, then $x^2 + y^2 \leq 1$. We have, by the triangle inequality,

$$\sqrt{x^2 + y^2} = |(x, y)| = |(x - x_0, y - y_0) + (x_0, y_0)| \leq 2\sqrt{\delta} + r,$$

where we used that if two points are in the same square of side length $\delta$, they are at distance at most $2\sqrt{\delta}$ from each other. Since $2\sqrt{\delta} + r < 1 - r + r = 1$, the claim is established.

Select now $k \in \mathbb{N}$ such that $2^{-k} < (1-r)^2/4$ and consider the set of all dyadic squares

$$Q_{j, \ell} = \left[ \frac{j}{2^k}, \frac{j + 1}{2^k} \right] \times \left[ \frac{\ell}{2^k}, \frac{\ell + 1}{2^k} \right], \quad (j, \ell) \in \mathbb{Z} \times \mathbb{Z}.$$

These squares clearly cover all of $\mathbb{R}^2$, and any two of them intersect each other only at the boundary (or not at all). Since they cover $\mathbb{R}^2$, it is clear that all those that intersect $D_r$ cover $D_r$. A formal, maybe pedantic, way of writing this is to define an index set, let’s call it $I$ by

$$I = \{(j, \ell) : Q_{j, \ell} \cap D_r \neq \emptyset\};$$
then obviously $D_r \subseteq \bigcup_{(j, \ell) \in I} Q_{j, \ell}$. I will rely here a lot on geometry and things that are geometrically obvious, such as that the set $I$ is finite and the conclusion that the total area of the squares indexed by $I$ has to exceed the area of $D_r$; i.e., $be \geq \pi r^2 > \pi (1 - \frac{\epsilon}{2}) = \pi - \epsilon$. On the other hand all of these squares will be contained in $D$, by the claim at the beginning of this argument.

4. Textbook, Exercise 26, Chapter 1, Page 45.

Solution.

(a) Assume $E$ is a convex subset of $\mathbb{R}^m$. We prove by induction on $n$: If $w_1, \ldots, w_n \in E$, $s_1, \ldots, s_n \in \mathbb{R}$, $s_k \geq 0$ for all $k$ and $s_1 + \cdots + s_n = 1$, then $s_1 w_1 + \cdots + s_n w_n \in E$. The case $n = 1$ is trivial and the case $n = 2$ is the definition of convexity. Assume thus the result proved for some $n \geq 2$ and let $w_1, \ldots, w_n, w_{n+1} \in E$, $0 \leq s_1, s_2, \ldots, s_{n+1}$, $s_1 + \cdots + s_{n+1} = 1$. To prove: $s_1 w_1 + \cdots + s_n w_{n+1} \in E$. We may assume that $s_k > 0$ for $k = 1, \ldots, n+1$; otherwise we are in the case of $n$ points. Then $\sigma := s_1 + \cdots + s_n > 0$; setting $t_k = s_k/\sigma$ for $k = 1, \ldots, n$, we see that $t_1, \ldots, t_n \geq 0$ and $t_1 + \cdots + t_n = 1$. By the induction hypothesis,

$$w := \frac{1}{\sigma}(s_1 w_1 + \cdots + s_n w_n) = t_1 w_1 + \cdots + t_n w_n \in E.$$

By the case $n = 2$ (convexity) since $\sigma + s_{n+1} = s_1 + \cdots + s_{n+1} = 1$,

$$s_1 w_1 + \cdots + s_n w_{n+1} = \sigma w + s_{n+1} w_{n+1} \in E.$$

(b) Too obvious to comment upon.

5. Textbook, Exercise 39, Chapter 1, Page 48.

(a) That uniform continuity implies continuity is almost too obvious to merit any attention. There are quite simple examples showing the continuity does not imply uniform continuity. The one proposed by the author of the textbook is not that simple. Here are first two very simple examples.

i. Consider $f(x) = 1/x$ for $x \in (0, 1)$. This function is clearly continuous. For a contradiction, assume it is absolutely continuous and let $\epsilon > 0$ be given. There should exist $\delta > 0$ such that if $x, y \in (0, 1)$ and $|x - y| < \delta$, then $|\frac{1}{x} - \frac{1}{y}| < \epsilon$. We may assume $\delta < 1$ (if any $\delta$ works, a smaller one works even better) and let $x \in (0, 1)$, $0 < x < 1/2$. Then taking $y = x + \delta/2$ we have $x, y \in (0, 1)$ and $|x - y| = y - x = \delta/2 < \delta$. Thus $|1/x - 1/y| < \epsilon$. But with this choice of $y$ we have $|1/x - 1/y| = \delta/(2xy) = \delta/|x(2x+\delta)| > \delta/2x$, since $2x+\delta < 2$. This proves: $\delta/2x < \epsilon$ for all $x \in (0,1/2)$, an inequality which becomes false the moment
Consider in the previous examples, uniform continuity was violated in a
The function \( f(x) = x^2 \) for \( x \in \mathbb{R} \). Then \(|x^2 - y^2| = |x + y||x - y|\)
and it is impossible to get \(|x^2 - y^2| < \epsilon\) for \(|x - y| < \delta\) without
controlling the size of \(|x + y|\). Given any \( \epsilon > 0 \), given any \( \delta > 0 \);
if we take \( \epsilon/2 \delta < x < y = x + \delta/2 \), then \(|x - y| < \delta\), but
\(|x^2 - y^2| > \epsilon\).

iii. In the previous examples, uniform continuity was violated in a
very strong way; there failed to exist \( \delta \) for EVERY \( \epsilon \). The text-
book example is not so drastic. To see that uniform continuity
is violated one has to take \( \epsilon \leq 1 \). Let \( f : (0, 1) \rightarrow \mathbb{R} \) be defined
by \( f(x) = \sin(1/x) \). Let \( \epsilon \in (0, 1) \) and assume there exists \( \delta > 0 \)
such that \( x, y \in (0, 1) \) and \(|x - y| < \delta\) implies \(|f(x) - f(y)| < \epsilon\).
With \( k \in \mathbb{N} \) let \( x = 2/[(4k + 1)\pi] \) and let \( y = 1/(2k\pi) \); \( k \) to be
determined. Obviously \( x, y \in (0, 1) \) and
\[
|x - y| = \left| \frac{2}{(4k + 1)\pi} - \frac{1}{2k\pi} \right| = \frac{1}{2k(4k + 1)\pi}.
\]
Clearly, by taking \( k \) large enough we can achieve that \( \frac{1}{2k(4k + 1)\pi} < \delta\); assume \( k \) so chosen. Then \(|x - y| < \delta\). But
\[
|f(x) - f(y)| = |\sin\left(\frac{4k + 1)\pi}{2}\right) - \sin(2k\pi)| = |1 - 0| = 1 > \epsilon.
\]
A contradiction has been reached; the function is not uniformly
continuous.

(b) The function \( f(x) = 2x \) is uniformly continuous in all of \( \mathbb{R} \). Let \( \epsilon > 0 \)
be given. Take \( \delta = \epsilon/2 \). Then \(|x - y| < \delta\) implies \(|f(x) - f(y)| = 2|x - y| < 2\delta = \epsilon\).

(c) The function \( f(x) = x^2 \) is not uniformly continuous in \( \mathbb{R} \). This was
proved above.


(a) If the sequence \( (a_n) \) of real numbers is bounded, then \( \{a_n : n \in \mathbb{N}\} \)
is bounded, hence it has a supremum \( \sigma \in \mathbb{R} \). Since \( \{a_k : k \geq n\} \subset \{a_n : n \in \mathbb{N}\} \) for all \( n \in \mathbb{N} \), for each \( n \in \mathbb{N} \), the set \( \{a_k : k \geq n\} \) is
bounded by \( \sigma \). Since it is obviously not empty, \( \sigma_n = \sup\{a_k : k \geq n\} \)
exists for each \( n \in \mathbb{N} \). These suprema are not necessarily equal
to \( \sigma \). Now comes the only not 100% trivial part of this exercise.
Claim: The sequence \( \{\sigma_n\} \) is decreasing and bounded below. In fact,
since \( \{a_k : k \geq n + 1\} \subset \{a_k : k \geq n\} \) for each \( n \in \mathbb{N} \), we get
\( \sigma_{n+1} \leq \sigma_n \). Since \( (a_n) \) is bounded, it is bounded below; say by \( a \).
Then, for each \( n, \sigma_n \geq a_n \geq a \). The claim is established. Since \( \{\sigma_n\} \) is decreasing and bounded below, it converges. It follows that 
\[ \lim \sup_{n \to \infty} a_n := \lim_{n \to \infty} a_n \] 
exists (in \( \mathbb{R} \)).

(b) Assume \( \sup \{ a_n : n \in \mathbb{N} \} = \infty \). Since we are dealing with a sequence of real numbers, and finite sets of real numbers are always bounded above, the only way this is possible is if we also have \( \sup \{ a_k : k \geq n \} = \infty \) for all \( n \). Thus the sequence of suprema is the sequence \( \{\sigma_n\} \) with \( \sigma_n = \infty \) for all \( n \), and \( \lim_{n \to \infty} \sigma_n = \infty \); thus \( \lim \sup_{n \to \infty} = \infty \) in this case.

(c) Assume \( \lim_{n \to \infty} a_n = -\infty \). The question on how to define \( \lim \sup_{n \to \infty} a_n \) is a bit strange: the question should be what is it? If we let \( \sigma_n = \sup \{ a_k : k \geq n \} \) for \( n \in \mathbb{N} \) as before, we get \( \lim_{n \to \infty} \sigma_n = -\infty \). In fact, let \( A \in \mathbb{R} \). Since \( \lim_{n \to \infty} a_n = -\infty \), there is \( N \) such that \( a_n < A \) for \( n \geq N \), implying easily that \( \sigma_n < A \) for \( n \geq N \). Thus \( \lim \sup_{n \to \infty} a_n = -\infty \) in this case.

(d) \[ \lim \sup_{n \to \infty} (a_n + b_n) \leq \lim_{n \to \infty} a_n + \lim \sup_{n \to \infty} b_n \] except if one of the two limsup's is \( \infty \) and the other one \( -\infty \). Assume first that one of \( \lim_{n \to \infty} a_n, \lim \sup_{n \to \infty} b_n \) equals \( \infty \). Since in this case we assume that the other one is \( > -\infty \), it follows that \( \lim_{n \to \infty} a_n + \lim \sup_{n \to \infty} b_n = \infty \), and the inequality is trivial. Assume now both are \( < \infty \). Let \( \sigma_n = \sup \{ a_k : k \geq n \} \), \( \tau_n = \sup \{ b_k : k \geq n \} \), for \( n \in \mathbb{N} \). If \( n \in \mathbb{N} \) and \( k \geq n \), then \( a_k \leq \sigma_n, b_k \leq \tau_n \), thus \( a_k + b_k \leq \sigma_n + \tau_n \). It follows that \( \sigma_n + \tau_n \) is an upper bound of \( \{ a_k + b_k : k \geq n \} \), hence \( \sup \{ a_k + b_k : k \geq n \} \leq \sigma_n + \tau_n \) for all \( n \in \mathbb{N} \) and

\[
\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} \sup \{ a_k + b_k : k \geq n \} \leq \lim_{n \to \infty} (\sigma_n + \tau_n) = \lim_{n \to \infty} \sigma_n + \lim_{n \to \infty} \tau_n = \lim_{n \to \infty} a_n + \lim \sup_{n \to \infty} b_n.
\]

(e) The \( \liminf \) of a sequence is defined briefly by

\[ \liminf a_n = \liminf_{n \to \infty} a_k. \]

The relation with \( \limsup \) is:

\[ \liminf_{n \to \infty} a_n = - \limsup_{n \to \infty} (-a_n). \]

I omit the simple proof.