LA Session Week 8 (MAC 2311)

Name:

1. Evaluate the derivatives of the following functions:

   (a) \( f(x) = \sin^{-1}(\ln x) \)
   (b) \( f(x) = \tan^{-1}(2x^2 - 4) \).

   \[ \frac{d}{dx} \sin^{-1}(\ln x) = \frac{1}{\sqrt{1 - (\ln x)^2}} \cdot \frac{1}{x} \]

   \[ \frac{d}{dx} \tan^{-1}(2x^2 - 4) = \frac{4x}{1 + (2x^2 - 4)^2} \]

2. The legs of an isosceles right triangle increase in length at a rate of 2 m/s.
(a). At what rate is the area of the triangle changing when the legs are 2 m long?
(b). At what rate is the area of the triangle changing when the hypotenuse is 1 m long?
(c). At what rate is the length of the hypotenuse changing?

Solution (a). Let \( x \) be the length of a leg of an isosceles right triangle, \( h \) be the length of the hypotenuse.

The area is \( A(x) = \frac{1}{2} x^2 \). It is given that \( \frac{dx}{dt} = 2 \text{ m/s} \).

Then, \( \frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt} = x \cdot \frac{dx}{dt} = 2x \text{ m}^2/\text{s} \).

When \( x = 2 \text{ m} \), we have \( \frac{dA}{dt} = 4 \text{ m}^2/\text{s} \), so the area is increasing at 4 square meters per second.

(b) when \( h = 1 \text{ m} \), then \( x^2 = \frac{1}{2} h^2 = \frac{1}{2} \) m \( x = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} \) m.

So, \( \frac{dA}{dt} = 2 \cdot \left( \frac{\sqrt{2}}{2} \right) = \sqrt{2} \text{ m}^2/\text{s} \).

(c) Since \( h^2 = x^2 + x^2 = 2x^2 \), it gives \( h = \sqrt{2} x \).

So, \( \frac{dh}{dt} = \sqrt{2} \cdot \frac{dx}{dt} = \sqrt{2} \cdot (2) = 2\sqrt{2} \text{ m/s} \).
3. Let \( f(x) = x^4 - 2x^3 + 1 \).
(a) Find the critical points of \( f \).
(b) Find the intervals on which \( f \) is increasing and decreasing.
(c) Find the local maximum and minimum values of \( f \).
(d) Find the intervals on which \( f \) is concave up or concave down.
(e) Identify any inflection points.

Solution. (a) \( f'(x) = 4x^3 - 6x^2 = 2x^2(2x-3) \). Solve \( f'(x) = 0 \), we have \( x = 0 \) and \( x = \frac{3}{2} \). So, \( x = 0 \) and \( x = \frac{3}{2} \) are the critical points of \( f \).

(b) Note that the two critical points divide the domain \((\infty, 0) \cup (0, \frac{3}{2}) \cup \left(\frac{3}{2}, \infty\right)\) into three subintervals \((-\infty, 0), (0, \frac{3}{2})\), and \(\left(\frac{3}{2}, \infty\right)\).

\[
\begin{array}{cccc}
\text{Interval} & f' & f' & f \\
(-\infty, 0) & + & - & - \quad \text{decreasing} \\
(0, \frac{3}{2}) & + & - & - \quad \text{decreasing} \\
\left(\frac{3}{2}, \infty\right) & + & + & + \quad \text{increasing} \\
\end{array}
\]

(c) \( f' \) is negative on both sides near \( x = 0 \), so \( f \) has no local extreme value at \( x = 0 \). \( f' \) changes sign from negative to positive as \( x \) increases through \( x = \frac{3}{2} \), so \( f \) has a local minimum.

Value of \( f'\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^4 - 2\left(\frac{3}{2}\right)^3 + 1 = -\frac{11}{16} \).

(d) \( f''(x) = 12x^2 - 12x = 12x(x-1) \). \( f''(x) = 0 \) at \( x = 0 \) and \( x = 1 \).

\[
\begin{array}{cccc}
\text{Interval} & f'' & f \\
(-\infty, 0) & - & + \quad \text{concave up} \\
(0, 1) & + & - \quad \text{concave down} \\
(1, \infty) & + & + \quad \text{concave up} \\
\end{array}
\]

(e) Since \( f'' \) changes sign at \( x = 0 \) and \( x = 1 \), there are inflection points at \( x = 0 \) and \( x = 1 \).