Summary of Chapter Five (Sections 5-1 and 5-4)

5-1. Two or More Random Variables

The random variables $X$ and $Y$ are said to be jointly continuous if there is a nonnegative function $f_{XY}(x, y)$, called the joint probability density function, such that

1. $f_{XY}(x, y) \geq 0$ for all $x, y$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y)dxdy = 1$
3. For any region $R$ in the $xy$-plane, $P((X, Y) \in R) = \int \int_{R} f_{XY}(x, y)dxdy$

The marginal density functions of $X$ and $Y$ are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx$$

If $X$ and $Y$ have a joint density function $f_{XY}(x, y)$, then

$$E[u(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y)f_{XY}(x, y)dxdy$$

Conditional Distributions

If $X$ and $Y$ are jointly continuous with joint density function $f_{XY}(x, y)$, then the conditional probability density function of $Y$ given that $X = x$ is given by

$$f_{Y|x}(y) = \frac{f_{XY}(x, y)}{f_X(x)}$$

and the conditional probability density function of $X$ given that $Y = y$ is given by

$$f_{X|y}(x) = \frac{f_{XY}(x, y)}{f_Y(y)}.$$

Some related formulas:

1. $P(a < X < b|Y = y) = \int_{a}^{b} f_{X|y}(x)dx$.
2. $P(a < Y < b|X = x) = \int_{a}^{b} f_{Y|x}(y)dy$. 

1
\( \mu_{Y|x} = E(Y|x) = \int_{-\infty}^{\infty} y f_{Y|x}(y) \, dy. \)
\( V(Y|x) = \int_{-\infty}^{\infty} [y - E(Y|x)]^2 f_{Y|x}(y) \, dx = \int_{-\infty}^{\infty} y^2 f_{Y|x}(y) \, dy - \mu_{Y|x}^2. \)

**Independent Random Variables**

The random variables \( X \) and \( Y \) are independent if for all sets \( A \) and \( B \)

\[ P(X \in A, Y \in B) = P(X \in A)P(Y \in B) \]

The independence of \( X \) and \( Y \) is equivalent to \( f_{XY}(x, y) = f_X(x)f_Y(y) \). If the joint density function factors into a part depending only on \( x \) and a part depending only on \( y \), then \( X \) and \( Y \) are independent.

**5-4. Linear Functions of Random Variables**

If \( X_1, X_2, \ldots, X_p \) are independent continuous random variables with respective p.d.f. \( f_{X_1}(x_1), f_{X_2}(x_2), \ldots, f_{X_p}(x_p) \), then the joint p.d.f. is

\[ f_{X_1X_2\ldots X_p}(x_1, x_2, \ldots, x_p) = f_{X_1}(x_1)f_{X_2}(x_2)\cdots f_{X_p}(x_p). \]

Given random variables \( X_1, X_2, \ldots, X_p \) and constants \( c_1, c_2, \ldots, c_p \),

\[ Y = c_1X_1 + c_2X_2 + \cdots + c_pX_p \]

is a linear combination of \( X_1, X_2, \ldots, X_p \).

Some related formulas:

1. \( E(Y) = c_1E(X_1) + c_2E(X_2) + \cdots + c_pE(X_p). \)
2. \( V(Y) = c_1^2V(X_1) + c_2^2V(X_2) + \cdots + c_p^2V(X_p) + 2 \sum_{i<j} c_ic_j\text{cov}(X_i, X_j), \)

where \( \text{cov}(X_i, X_j) = E[(X_i - E(X_i))(X_j - E(X_j))] \). If \( X_1, X_2, \ldots, X_p \) are independent, then

\[ V(Y) = c_1^2V(X_1) + c_2^2V(X_2) + \cdots + c_p^2V(X_p). \]

3. For sample mean \( \bar{X} = (X_1 + X_2 + \cdots + X_p)/p \) with \( E(X_i) = \mu \) for \( i = 1, 2, \ldots, p \), we have \( E(\bar{X}) = \mu \). If \( X_1, X_2, \ldots, X_p \) are also independent with \( V(X_i) = \sigma^2 \) for \( i = 1, 2, \ldots, p \), we have

\[ V(\bar{X}) = \frac{\sigma^2}{p}. \]
(4) If $X_1, X_2, \ldots, X_p$ are independent, normal random variables with $E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2$ for $i = 1, 2, \ldots, p$, then $Y = c_1X_1 + c_2X_2 + \cdots + c_pX_p$ is a normal random variable with

\[
E(Y) = c_1\mu_1 + c_2\mu_2 + \cdots + c_p\mu_p
\]

and

\[
V(Y) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_p^2\sigma_p^2.
\]