Summary of Chapters 7 and 8

Chapter 7. Sampling Distributions and Point Estimation of Parameters

If \( X_1, \cdots, X_n \) is a random sample from a population with normal distribution \( N(\mu, \sigma^2) \), then the sample mean \( \bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n} \) has a normal distribution \( N(\mu, \frac{\sigma^2}{n}) \).

Central Limit Theorem:
If \( X_1, \cdots, X_n \) is a random sample of size \( n \) taken from a population (either finite or infinite) with mean \( \mu \) and finite variance \( \sigma^2 \), and if \( \bar{X} \) is the sample mean, the limiting distribution of
\[
Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}
\]
is the standard normal distribution as \( n \to \infty \).

Unbiased Estimators:
The point estimator \( \hat{\Theta} \) is an unbiased estimator for the parameter \( \theta \) if \( E(\hat{\Theta}) = \theta \). If the estimator is not unbiased, then the difference \( E(\hat{\Theta}) - \theta \) is called the bias of the estimator \( \hat{\Theta} \).

The mean squared error of an estimator \( \hat{\Theta} \) of the parameter \( \theta \) is defined as
\[
MSE(\hat{\Theta}) = E(\hat{\Theta} - \theta)^2 = V(\hat{\Theta}) + (bias)^2,
\]
where \( V(\hat{\Theta}) \) is the variance of \( \hat{\Theta} \). For two estimator \( \hat{\Theta}_1, \hat{\Theta}_2 \), the relative efficiency of \( \hat{\Theta}_2 \) to \( \hat{\Theta}_1 \) is defined as
\[
\frac{MSE(\hat{\Theta}_1)}{MSE(\hat{\Theta}_2)}.
\]
If this relative efficiency is less than 1, then \( \hat{\Theta}_1 \) is a more efficient (better) estimator of \( \theta \) than \( \hat{\Theta}_2 \).

Methods of Moments:
Let \( X_1, X_2, \cdots, X_n \) be a random sample from the probability distribution \( f(x) \) (pdf or pmf). The \( k \)th population moment is \( E(X^k), k = 1, 2, \ldots \). The corresponding \( k \)th sample moment is
\[
\frac{1}{n} \sum_{i=1}^{n} X_i^k, \quad k = 1, 2, \ldots
\]
Let the population distribution have \( m \) unknown parameters \( \theta_1, \theta_2, \ldots, \theta_m \). the moment estimators \( \hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_m \) are found by equating the first \( m \) population moments to the first \( m \) sample moments and solving the resulting equations for the unknown parameters.

Maximum Likelihood Estimator:
Suppose that \( X \) is a random variable with probability distribution \( f(x; \theta) \), where \( \theta \) is a single unknown parameter. Let \( x_1, x_2, \cdots, x_n \) be the observed values in a random sample of size \( n \). Then the likelihood function of the sample is
\[
L(\theta) = f(x_1; \theta) \cdot f(x_2; \theta) \cdots f(x_n; \theta)
\]
The maximum likelihood estimator (MLE) of \( \theta \) is the value of \( \theta \) that maximizes the likelihood function \( L(\theta) \). Basically, we solve the equation
\[
\frac{d \ln L(\theta)}{d \theta} = 0
\]
and its solution gives the MLE of $\theta$.

Chapter 8. Statistical Intervals for a Single Sample

Confidence interval for $\mu$ when $\sigma$ is known:
If $\bar{x}$ is the sample mean of a random sample of size $n$ from a normal population with known variance $\sigma^2$, a 100$(1 - \alpha)$% CI on $\mu$ is given by

$$\bar{x} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}},$$

where $z_{\alpha/2}$ is the upper 100$\alpha/2$ percentage point of the standard normal distribution.

If $\bar{x}$ is used as an estimate of $\mu$, we can be 100$(1 - \alpha)$% confident that the error $|\bar{x} - \mu|$ will not exceed a specified amount $E$ when the sample size is

$$n = \left( \frac{z_{\alpha/2} \sigma}{E} \right)^2$$

A 100$(1 - \alpha)$% upper-confidence bound for $\mu$ is

$$\mu \leq u = \bar{x} + z_{\alpha} \frac{\sigma}{\sqrt{n}}$$

and a 100$(1 - \alpha)$% lower-confidence bound for $\mu$ is

$$\bar{x} - z_{\alpha} \frac{\sigma}{\sqrt{n}} = l \leq \mu$$

Confidence interval on $\mu$ when $\sigma^2$ is unknown:
Let $X_1, \ldots, X_n$ be a random sample from a normal distribution $N(\mu, \sigma^2)$ with unknown $\mu$ and $\sigma^2$. The random variable

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has a $t$ distribution with $n - 1$ degrees of freedom.

A 100$(1 - \alpha)$% CI on $\mu$ (with unknown $\sigma^2$) is given by

$$\bar{x} - t_{\alpha/2,n-1} \frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{\alpha/2,n-1} \frac{s}{\sqrt{n}},$$

where $t_{\alpha/2,n-1}$ is the upper 100$\alpha/2$ percentage point of the $t$ distribution with $n - 1$ degrees of freedom.

A 100$(1 - \alpha)$% upper-confidence bound for $\mu$ is

$$\mu \leq u = \bar{x} + t_{\alpha,n-1} \frac{s}{\sqrt{n}}$$

and a 100$(1 - \alpha)$% lower-confidence bound for $\mu$ is

$$\bar{x} - t_{\alpha,n-1} \frac{s}{\sqrt{n}} = l \leq \mu$$
CI for population proportion $p$:
If $\hat{p}$ is the proportion of observations in a random sample of size $n$ that belongs to a class of interest, an approximate $100(1 - \alpha)%$ confidence interval on $p$ is

$$\hat{p} - z_{\alpha/2} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} \leq p \leq \hat{p} + z_{\alpha/2} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution.

The Choice of Sample Size:
$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 \hat{p}(1 - \hat{p})$$

The approximate $100(1 - \alpha)%$ lower and upper confidence bounds are

$$\hat{p} - z_{\alpha} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} \leq p \text{ and } p \leq \hat{p} + z_{\alpha} \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$$

Prediction and Tolerance Interval:
A $100(1 - \alpha)%$ prediction interval on a single future observation from a normal distribution is given by

$$\bar{x} - t_{\alpha/2,n-1} s \sqrt{1 + \frac{1}{n}} \leq X_{n+1} \leq \bar{x} + t_{\alpha/2,n-1} s \sqrt{1 + \frac{1}{n}}$$

A tolerance interval for capturing at least $100\gamma%$ of the values in a normal distribution with confidence level $100(1 - \alpha)%$ is

$$\bar{x} - ks, \bar{x} + ks$$

where $k$ is a tolerance interval factor found in Appendix Table XII. Values are given for $\gamma = 90\%, 95\%,$ and $99\%$ and for $90\%, 95\%$ and $99\%$ confidence.