Solutions to Quiz # 4 (STA 4032)

1. Let $X$ and $Y$ have a joint probability density function

$$f_{XY}(x, y) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq x \leq y \leq 2, \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the marginal pdfs of $X$ and $Y$.
(b) Are $X$ and $Y$ independent? Explain your answer.
(c) Find $f_{Y|x}(y)$ and $E(Y|x)$.
(d) Compute $P(1 \leq Y \leq \frac{3}{2} | X = 1)$.

Solution. (a) By definition, the marginal pdf of $X$ is

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy = \int_x^2 \frac{1}{2} dy = \frac{2 - x}{2}, \quad 0 \leq x \leq 2.$$ 

Otherwise, $f_X(x) = 0$. Similarly, we have

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx = \int_y^2 \frac{1}{2} dx = \frac{y}{2}, \quad 0 \leq y \leq 2.$$ 

Otherwise, $f_Y(y) = 0$.

(b) $X$ and $Y$ are not independent, since $f_{XY}(x, y) \neq f_X(x)f_Y(y)$.

(c) By definition, we have

$$f_{Y|x}(y) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{\frac{1}{2}}{\frac{2-x}{2}} = \frac{1}{2-x}, \quad 0 \leq x \leq y \leq 2$$

and

$$E(Y|x) = \int_{-\infty}^{\infty} y f_{Y|x}(y) dy = \int_x^2 y \cdot \frac{1}{2-x} dy = \left[ \frac{y^2}{2(2-x)} \right]_x^{\infty} = \frac{2 + x}{2}, \quad 0 \leq x \leq 2.$$

(d) 

$$P \left( 1 \leq Y \leq \frac{3}{2} | X = 1 \right) = \int_1^{3/2} f_{Y|1}(y) dy = \int_1^{3/2} \frac{1}{2-1} dy = 1/2.$$ 

2. Soft-drink cans are filled by an automated filling machine. Assume that the fill volumes of the cans are independent and normally distributed with mean fill volume 12.1 ounces and standard deviation 0.5 ounces.

(a) What is the standard deviation of the sample mean fill volume of 100 cans?
(b) What is the probability that the average fill volume of the 100 cans is below 12
fluid ounces?
(c) Determine the number of cans that need to be measured such that the probability that the average fill volume is less than 12 fluid ounces is 0.01.

**Solution.** (a) Let $\bar{X}$ be the sample mean fill volume of 100 cans. Then, $\bar{X}$ has a normal distribution $N(\mu, \sigma^2/n) = N(12.1, 0.5^2/100)$. So the standard deviation of $\bar{X}$ is $\sigma_{\bar{X}} = \sqrt{0.5^2/100} = 0.5/10 = 0.05$ ounces.

(b) The desired probability is

$$P(\bar{X} < 12) = P(Z < \frac{12 - 12.1}{0.05}) = P(Z < -2) = 0.023.$$ 

(c) $P(\bar{X} < 12) = 0.01$ implies that $P(Z < \frac{12 - 12.1}{0.5/\sqrt{n}}) = 0.01$. Then $\frac{12 - 12.1}{0.5/\sqrt{n}} = -2.33$ and $n = 135.72 \approx 136$. 