Chapter 4. Continuous Distributions

Let \( X \) be a continuous random variable with probability density function \( f(x) \). The *cumulative distribution function* of \( X \) is defined by

\[
F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y) \, dy.
\]

We have the following formulae:

1. \( \int_{-\infty}^{\infty} f(x) \, dx = 1. \)
2. \( P(a < X \leq b) = \int_{a}^{b} f(x) \, dx = F(b) - F(a). \)
3. \( \mu = E(X) = \int_{-\infty}^{\infty} x f(x) \, dx. \)
4. \( \sigma^2 = V(X) = E[(X - E(X))^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx \)
5. \( \sigma^2 = E(X^2) - \mu^2. \)
6. For any function \( h \),
   \[
   E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x) \, dx.
   \]

Some Special Continuous Distributions:

1. Uniform distribution: p.d.f. \( f(x) = 1/(b - a), \; a \leq x \leq b; \) Otherwise, \( f(x) = 0. \)
2. Let \( X \) have a normal distribution \( N(\mu, \sigma^2) \). Then, \( E[X] = \mu \) and \( Var(X) = \sigma^2 \). Let \( Z = (X - \mu)/\sigma \). Then, \( Z \) has a standard normal distribution \( N(0, 1) \) with \( \mu = 0 \) and \( \sigma = 1 \). The distribution function of \( X \) can be expressed by
   \[
   F_X(a) = P(X \leq a) = P\left( \frac{X - \mu}{\sigma} \leq \frac{a - \mu}{\sigma} \right) = P\left( Z \leq \frac{a - \mu}{\sigma} \right) = \Phi\left( \frac{a - \mu}{\sigma} \right),
   \]
   where \( \Phi \) is the distribution function of \( Z \). We also have
   \[
   P(a \leq X \leq b) = F(b) - F(a) = \Phi\left( \frac{b - \mu}{\sigma} \right) - \Phi\left( \frac{a - \mu}{\sigma} \right).
   \]
3. Exponential distribution with p.d.f. \( f(x) = \lambda e^{-\lambda x}, \; 0 \leq x < \infty \; (\lambda > 0) \). The cumulative probability distribution function is given by
   \[
   F(x) = \begin{cases} 
   1 - e^{-\lambda x}, & x \geq 0 \\
   0, & \text{otherwise}
   \end{cases}
   \]
   The expected value and variance of \( X \) are
   \[
   E(X) = \frac{1}{\lambda}, \quad V(X) = \frac{1}{\lambda^2}.
   \]
It satisfies the memoryless property, for positive $s$ and $t$,

$$P(X > s + t | X > t) = P(X > s).$$

Chapter 5. Joint Probability Distributions (Sections 5-2 and 5-5)

5-2. Two Continuous Random Variables

The random variables $X$ and $Y$ are said to be jointly continuous if there is a nonnegative function $f_{XY}(x, y)$, called the joint probability density function, such that

1. $f_{XY}(x, y) \geq 0$ for all $x, y$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1$
3. For any region $R$ in the $xy$-plane,

$$P((X, Y) \in R) = \int \int_R f_{XY}(x, y) dx dy$$

The marginal density functions of $X$ and $Y$ are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) dx$$

Conditional Distributions

If $X$ and $Y$ are jointly continuous with joint density function $f_{XY}(x, y)$, then the conditional probability density function of $Y$ given that $X = x$ is given by

$$f_{Y|x}(y) = \frac{f_{XY}(x, y)}{f_X(x)}$$

and the conditional probability density function of $X$ given that $Y = y$ is given by

$$f_{X|y}(x) = \frac{f_{XY}(x, y)}{f_Y(y)}.$$ 

Some related formulas:

1. $P(a < X < b | Y = y) = \int_a^b f_{X|y}(x) dx$.
2. $P(a < Y < b | X = x) = \int_a^b f_{Y|x}(y) dy$.
3. $\mu_{Y|x} = E(Y | x) = \int_{-\infty}^{\infty} y f_{Y|x}(y) dy$.
4. $V(Y | x) = \int_{-\infty}^{\infty} [y - E(Y | x)]^2 f_{Y|x}(y) dy = \int_{-\infty}^{\infty} y^2 f_{Y|x}(y) dy - \mu_{Y|x}^2$.

Independent Random Variables

Two continuous random variables $X$ and $Y$ are independent if $f_{XY}(x, y) = f_X(x)f_Y(y)$.

5-5. Linear Functions of Random Variables

Given random variables $X_1, X_2, \ldots, X_p$ and constants $c_1, c_2, \ldots, c_p$,

$$Y = c_1 X_1 + c_2 X_2 + \cdots + c_p X_p$$
is a linear combination of $X_1, X_2, \ldots, X_p$.

Some related formulas:

(1) $E(Y) = c_1 E(X_1) + c_2 E(X_2) + \cdots + c_p E(X_p)$.

(2) If $X_1, X_2, \ldots, X_p$ are independent, then

$$V(Y) = c_1^2 V(X_1) + c_2^2 V(X_2) + \cdots + c_p^2 V(X_p).$$

(3) If $X_1, X_2, \ldots, X_p$ are independent, normal random variables with $E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2$ for $i = 1, 2, \ldots, p$, then $Y = c_1 X_1 + c_2 X_2 + \cdots + c_p X_p$ is a normal random variable with

$$E(Y) = c_1 \mu_1 + c_2 \mu_2 + \cdots + c_p \mu_p$$

and

$$V(Y) = c_1^2 \sigma_1^2 + c_2^2 \sigma_2^2 + \cdots + c_p^2 \sigma_p^2.$$