(1) (15 pts.) Complete the following definitions.
   (a) If $G$ is a group and $H$ is a non-empty subset of $G$, then we say $H$ is a subgroup of $G$ if . . .
   (b) If $G$ is a group and $x \in G$, the cyclic subgroup generated by $x$ is . . .
   (c) A homomorphism of rings is . . .

(2) (10 pts.) Let $G$ be a finite group and $x \in G$. Prove: If $H$ is a subgroup of $G$, then the sets $xH$ and $H$ have the same number of elements.

(3) (10 pts.) Prove: If $G$ is a finite abelian group of order $n$ and $x \in G$, then $x^n = e$.

(4) (10 pts.) Give an example of each of the following. Proofs are not required, nevertheless, be specific and describe your examples carefully. If an example does not exist, say so.
   (a) A field of characteristic zero.
   (b) A field of characteristic 21.
   (c) A field of characteristic 23.
   (d) An abelian group of order 99.
   (e) A non-abelian group.
   (f) A group of order 7 containing an element of order 2.

(5) (15 pts.)
   (a) Compute the order of 225 in $\mathbb{Z}/14$.
   (b) Compute the order of 225 in $\mathbb{Z}/11$.
   (c) Use the answers in parts (a) and (b) to compute the order of 225 in $\mathbb{Z}/154$.
   (d) Use the answer in part (c) to find the remainder when $(225)^{241}$ is divided by 154.

(6) (20 pts.) Let $R$ denote the ring $\text{Mat}_2(\mathbb{Z}/2)$ of two-by-two matrices over $\mathbb{Z}/2$.
   (a) What is the characteristic of $R$?
   (b) For each $x \in R$, compute $x^2$.
   (c) List the group of units of $R$. For each unit $x$, find the order of $x$.
   (d) True or false? If $x$ has order 2 and $y$ has order 2, then $xy$ has order 2.
   (e) List the set of zero divisors of $R$.

(7) (20 pts.) Let $G = U_{16}$ be the group of units in the ring $\mathbb{Z}/16$.
   (a) List the elements in $G$.
   (b) Compute the order of each element of $G$.
   (c) True or false? If $x$ has order 2 and $y$ has order 2, then $xy$ has order 2.
   (d) Let $H$ be the subset of $G$ consisting of all elements of order 1 or 2. Show that $H$ is a subgroup of $G$. 