(1) (20 points) Complete the following definitions.
   (a) Let $G$ be a group and $H$ a subset of $G$. We say $H$ is a **subgroup** if . . . .
   (b) Let $R$ and $S$ be rings. A **homomorphism** from $R$ to $S$ is . . . .
   (c) If $G$ is a group, $a$ is an element of $G$, and $d > 0$, we say $a$ has order $d$ if . . . .
   (d) Let $R$ be a ring. The **characteristic** of $R$ is . . . .

(2) (15 points) Let $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8. Answer the following:
   (a) For each $x \in G$, find the order of $x$.
   (b) Find a subgroup $H$ of order 2 and find the partition of $G$ into left cosets of $H$.
   (c) Let $H = \langle i \rangle$ be the subgroup generated by $i$. Find the partition of $G$ into left cosets of $H$.

(3) (10 points) Let $G$ be an abelian group and $a$ and $b$ elements of $G$. Assume $|a| = 2$ and $|b| = 3$.
   (a) List the elements in the subgroup $\langle ab \rangle$.
   (b) Find the order of the element $ab$.

(4) (10 points) Give an example for each of the following. If none exists, say so.
   (a) A field of order 4.
   (b) A field of characteristic 4.
   (c) A field of characteristic 0.
   (d) A ring of order 4 which is not a field.
   (e) A non-commutative ring of characteristic 2.
   (f) An abelian group of order 144.

(5) (20 points) Let $G$ be a finite group and $x \in G$. Prove the following.
   (a) $x^n = e$ for some positive integer $n$.
   (b) The order of $x$ is equal to the number of elements in the subgroup generated by $x$.

(6) (10 points) Let $G$ be a finite group, $H$ a subgroup of $G$, and $x \in G$. Prove the following.
   (a) The left coset $xH$ has the same number of elements as $H$.
   (b) The sets $xG$ and $G$ are equal.

(7) (15 points) Let $p$ be a prime number and $G$ a group of order $p$.
   (a) Use Lagrange’s Theorem to prove that the order of any $x$ in $G$ is equal to 1 or $p$.
   (b) Use part (a) to show that a ring $R$ of order $p$ has characteristic $p$. 