Vectors & Vector Diagram
What are Vectors?

• Vectors are those physical quantities which have both magnitude and direction. We usually represent a vector with an arrow:

• Graphically, the direction of the arrow is the direction of the vector, the length is the magnitude.
  - Example. Force, velocity, displacement, acceleration

• Quantitatively: A vector $\mathbf{v}$ in the Cartesian plane is an ordered pair of real numbers that has the form $<a, b>$ or $(a, b)$. We write $\mathbf{v}=<a, b>$ where $a$ and $b$ are the components of vector $\mathbf{v}$. 
Multiples of Vectors

Given a real number $c$, we can multiply a vector by $c$ by multiplying its magnitude by $c$:

$v$ \quad 2v \quad -2v$

Notice that multiplying a vector by a negative real number reverses the direction.
Scalar Multiplication of vectors

Here $w=2v$, $x=(1/2)v$, and $w=4x$ but $v_2 = (-2/3)V$. 

$v$ 300 mph
$w$ 600 mph
$x$ 150 mph
$v_2$ 200 mph
Operation with Vectors

Two vectors can be added using the **Parallelogram Law**

\[ \mathbf{u} + \mathbf{v} \]

The sum of two vectors, \( \mathbf{u} = \langle u_1, u_2 \rangle \) and \( \mathbf{v} = \langle v_1, v_2 \rangle \) is the vector

\[ \mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle. \]

For example, if \( \mathbf{u} = (-2, 3) \) and \( \mathbf{v} = (4, 1) \), then

\[ \mathbf{u} + \mathbf{v} = (-2 + 4, 3 + 1) = (2, 4) \]

\[ \mathbf{u} - 2\mathbf{v} = (-2, 3) - 2(4, 1) = (-2, 3) - (8, 2) = (-2 - 8, 3 - 2) = (-10, 1). \]
Position Vectors

For a vector $v$ with initial point $(x_1, y_1)$ and terminal point $(x_2, y_2)$, the position vector for $v$ is

$$v = (x_2 - x_1, y_2, -y_1)$$

an equivalent vector with initial point $(0, 0)$ and terminal point $(x_2 - x_1, y_2, -y_1)$.

• The magnitude of a vector (length) of $v = (a, b)$ is found by using the Pythagorean theorem: $|v| = |(a, b)| = \sqrt{a^2 + b^2}$

Question: Vector $v = (12, -5)$ has initial point $(-4, 3)$.
   a) Find the coordinates of the terminal point.
   b) Find the length of the vector.
To do computations with vectors, we place them in the plane and find their components.

The initial point is the tail, the head is the terminal point. The components are obtained by subtracting coordinates of the initial point from those of the terminal point.

The first component of \( \mathbf{v} \) is \( 5 - 2 = 3 \) and the second is \( 6 - 2 = 4 \).

We write \( \mathbf{v} = <3,4> \)
Unit Vectors

A unit vector is a vector with magnitude 1.

Given a vector \( \mathbf{v} \), we can form a unit vector by multiplying the vector by \( 1/||\mathbf{v}|| \).

For example, find the unit vector in the direction \( \mathbf{a} = \langle 3, -4 \rangle \).

If \( \mathbf{a} = \langle 3, -4 \rangle \), then \( ||\mathbf{a}|| = \sqrt{a^2 + b^2} = 5 \). So, \( \langle 3/5, -4/5 \rangle \) is a unit vector in the same direction as \( \mathbf{a} \).

Special Vectors

A vector such as \( \langle a, b \rangle \) can be written as \( a\langle 1, 0 \rangle + b\langle 0, 1 \rangle \).

For this reason, these vectors are given special names: \( \mathbf{i} = \langle 1, 0 \rangle \) and \( \mathbf{j} = \langle 0, 1 \rangle \). A vector in component form \( \mathbf{v} = \langle a, b \rangle \) can be written \( a\mathbf{i} + b\mathbf{j} \).
Example

Question: Let $\mathbf{P} = (2, -3)$ and $\mathbf{Q} = (-5, 7)$. Find the unit vector along the direction of $\mathbf{PQ}$.

Answer:

Here $\mathbf{P} = (2, -3)$ and $\mathbf{Q} = (-5, 7)$.

So vector $\mathbf{PQ} = \mathbf{Q} - \mathbf{P} = (-5, 7) - (2, -3) = (-7, 10)$

Now $|\mathbf{PQ}| = \sqrt{(-7)^2 + (10)^2} = \sqrt{49 + 100} = \sqrt{149}$

So the unit vector along $\mathbf{PQ}$ is $\frac{\mathbf{PQ}}{|\mathbf{PQ}|} = \frac{(-7, 10)}{\sqrt{149}} = (-\frac{7}{\sqrt{149}}, \frac{10}{\sqrt{149}})$
The dot product of \( \mathbf{u} = \langle u_1, u_2 \rangle \) and \( \mathbf{v} = \langle v_1, v_2 \rangle \) is
\[
\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2
\]

Examples: Find each dot product.

a. \( \langle 4,5 \rangle \cdot \langle 2,3 \rangle = 4(2) + 5(3) = 23 \)

b. \( \langle 2,-1 \rangle \cdot \langle 1,2 \rangle = 2(1) + (-1)(2) = 0 \)

c. \( \langle 0,3 \rangle \cdot \langle 4,-2 \rangle = 0(4) + 3(-2) = -6 \)
The Angle Between Two Vectors

If \( \theta \) is the angle between two nonzero vectors \( u \) and \( v \), then

\[
\cos \theta = \frac{u \cdot v}{\|u\|\|v\|}
\]

Find the angle between

\( u = \langle 4, 3 \rangle \) & \( v = \langle 3, 5 \rangle \)

\[
\cos \theta = \frac{\langle 4, 3 \rangle \cdot \langle 3, 5 \rangle}{\|\langle 4, 3 \rangle\|\|\langle 3, 5 \rangle\|} = \frac{27}{5\sqrt{34}}
\]

\[
\theta = \arccos \frac{27}{5\sqrt{34}} \approx 22.2^\circ
\]
Special Cases:

Case I: Parallel Vectors ($\theta = 0$ degree)
If two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel then $\cos \theta = \cos 0 = 1$
Thus $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|$
More precisely, $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u} = m\mathbf{v}$ for some scalar $m$.

Case II: Orthogonal Vector ($\theta = 90$ degree)
If two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal then $\cos \theta = \cos 90 = 0$
Thus $\mathbf{u} \cdot \mathbf{v} = 0$

Question: Are the vectors $\mathbf{u} = \langle 2, -3 \rangle$ & $\mathbf{v} = \langle 6, 4 \rangle$ orthogonal?

Solution: Find the dot product of the two vectors.
$\mathbf{u} \cdot \mathbf{v} = \langle 2, -3 \rangle \cdot \langle 6, 4 \rangle = 2(6) + (-3)(4) = 0$

Because the dot product is 0, the two vectors are orthogonal.