The Circles of Lester, Evans, Parry, and Their Generalizations

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Abstract: Beginning with the famous Lester circle containing the circumcenter, nine-point center and the two Fermat points of a triangle, we survey a number of interesting circles in triangle geometry.
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1. Some common triangle centers

Figure 1. The Euler line and the nine-point circle

\[HN : NG : GO = 3 : 1 : 2.\]
Figure 2. The orthocentroidal circle

$O$ and $N$ are inverse in the orthocentroidal circle.
The reflections of the Euler line in the three sidelines intersect at a point on the circumcircle:

\[ E = \left( \frac{a^2}{b^2 - c^2} : \frac{b^2}{c^2 - a^2} : \frac{c^2}{a^2 - b^2} \right). \]
Construct equilateral triangles $A'B'C$, $AB'C$, $ABC'$ externally on the sides of triangle $ABC$.

$AA'$, $BB'$, and $CC'$ concur at the **Fermat point**

$$F_+ = K \left( \frac{\pi}{3} \right) = \left( \frac{1}{\sqrt{3}S_A + S} : \frac{1}{\sqrt{3}S_B + S} : \frac{1}{\sqrt{3}S_C - S} \right).$$
If the equilateral triangles $A''BC$, $AB''C$, $ABC''$ are constructed internally, $AA''$, $BB''$, and $CC''$ concur at the **negative Fermat point**

$$F_\_ = K \left( -\frac{\pi}{3} \right) = \left( \frac{1}{\sqrt{3}S_A - S} : \frac{1}{\sqrt{3}S_B - S} : \frac{1}{\sqrt{3}S_C + S} \right).$$
Kiepert triangle $\mathcal{K}(\theta) := XYZ$,

Kiepert perspector $K(\theta) = \left( \frac{1}{S_A + S_\theta} : \frac{1}{S_B + S_\theta} : \frac{1}{S_C + S_\theta} \right)$. 

Figure 6. Kiepert triangle $\mathcal{K}(\theta)$ and Kiepert perspector $K(\theta)$
The locus of the Kiepert perspector is a rectangular hyperbola whose center is the midpoint of the Fermat points.

\[(b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy = 0.\]
2. The first Lester circle

Theorem 1 (Lester). *The Fermat points are concyclic with the circumcenter and the nine-point center.*

![Figure 8](image-url)
Proof. (1) Let $M$ be the intersection of $F_+F_-$ and the Euler line. By the intersection chords theorem, it is enough to show that

$$MF_+ \cdot MF_- = MO \cdot MN.$$
(2) Consider a Kiepert perspector \( K(\theta) \) with homogeneous barycentric coordinates

\[
K(\theta) = \left( \frac{1}{S_A + S_\theta} : \frac{1}{S_B + S_\theta} : \frac{1}{S_C + S_\theta} \right).
\]

These homogeneous coordinates can be rewritten as

\[
K(\theta) = \left( (S_B + S_\theta)(S_C + S_\theta), (S_C + S_\theta)(S_A + S_\theta), (S_A + S_\theta)(S_B + S_\theta) \right)
= (S_{BC} + S_{\theta\theta} + (S_B + S_C)S_\theta, \ldots, \ldots)
= (S_{BC} + S_{\theta\theta}, S_{CA} + S_{\theta\theta}, S_{AB} + S_{\theta\theta} + S_\theta(S_B + S_C, S_C + S_A, S_A + S_B)).
\]

Similarly,

\[
K(-\theta) = (S_{BC} + S_{\theta\theta}, S_{CA} + S_{\theta\theta}, S_{AB} + S_{\theta\theta}) - S_\theta(S_B + S_C, S_C + S_A, S_A + S_B).
\]

From these, \( K(\theta) \) and \( K(-\theta) \) divide harmonically the symmedian point \( K = (S_B + S_C, S_C + S_A, S_A + S_B) \) and

\[
Q(\theta) = (S_{BC} + S_{\theta\theta}, S_{CA} + S_{\theta\theta}, S_{AB} + S_{\theta\theta})
= (S_{BC}, S_{CA}, S_{AB}) + S_{\theta\theta}(1, 1, 1)
\]

which is a point on the Euler line, dividing the orthocenter \( H = (S_{BC}, S_{CA}, S_{AB}) \) and the centroid \( G = (1, 1, 1) \) in the ratio

\[
GQ(\theta) : Q(\theta)H = 3S_{\theta\theta} : S^2 = 3\cot^2\theta : 1.
\]
\[ GQ(\theta) : Q(\theta)H = 3 S_{\theta \theta} : S^2 = 3 \cot^2 \theta : 1. \]
(3) For $\theta = \pm \frac{\pi}{3}$, this ratio is $1 : 1$.

$M = Q \left( \frac{\pi}{3} \right)$ is the midpoint of $GH$.

The Fermat line $F_+F_-$ intersects the Euler line at the midpoint of $H$ and $G$, which is the center of the **orthocentroidal circle** with $HG$ as diameter.

![Figure 11. Fermat line and orthocentroidal circle](image-url)
(4) If we put $OH = 6d$, then

$$MO \cdot MN = 4d \cdot d = (2d)^2 = (MH)^2 = (MG)^2.$$  

(5) Recall from (1) $MF_+ \cdot MF_- = MO \cdot MN = (MH)^2 = (MG)^2$. 

![Figure 12. The Euler line](image1)

![Figure 13. Fermat line and orthocentroidal circle](image2)
Therefore, the Lester circle theorem is equivalent to each of the following.

1. The Fermat points are inverse in the orthocentroidal circle.
2. The circle $F_+F_-G$ is tangent to the Euler line at $G$.
3. The circle $F_+F_-H$ is tangent to the Euler line at $H$.

![Figure 14](image-url) The circles $F_+F_-G$ and $F_+F_-H$
**Theorem 2.** The Fermat points are inverse in the orthocentroidal circle.

![Figure 15.](image)

**Proof.** Let $M$ be the matrix of the orthocentroidal circle.

$$M = \begin{pmatrix}
-4S_A & S_A + S_B & S_A + S_C \\
S_A + S_B & -4S_B & S_B + S_C \\
S_A + S_C & S_B + S_C & -4S_C
\end{pmatrix}.$$  

Write

$$F_+ = X + Y \quad \text{and} \quad F_- = X - Y,$$
with
\[ X = (S_{BC} + \frac{1}{3}S^2 \ S_{CA} + \frac{1}{3}S^2 \ S_{AB} + \frac{1}{3}S^2), \]
\[ Y = \frac{S}{\sqrt{3}} (S_B + S_C \ S_C + S_A \ S_A + S_B). \]

\[ XMX^t = YMY^t = \frac{2}{3} (S_A(S_B - S_C)^2 + S_B(S_C - S_A)^2 + S_C(S_A - S_B)^2)S^2, \]
we have
\[ F_+MF_-^t = (X + Y)M(X - Y)^t = XMX^t - YMY^t = 0. \]

This shows that the Fermat points are inverse in the orthocentroidal circle.

**Corollary 3.** Every circle through \( F_+ \) and \( F_- \) is orthogonal to the orthocentroidal circle.
**Theorem 4** (Gibert). *Every circle with diameter a chord of the Kiepert hyperbola perpendicular to the Euler line passes through the Fermat points.*
Equation of line $F_+F_-$: $L = 0$.
Perpendicular to Euler line at $H$: $L_0 = 0$,
intersecting Kiepert hyperbola at $Y_0$.
Circle $F_+F_-H$ is one in the pencil of conics through $F_+, F_-, H$ and $Y_0$:
\[(b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy - L \cdot L_0 = 0\]
by suitably adjusting the linear forms $L, L_0$ by constants.
It center lies on the perpendicular bisector of $F_+F_-$. 
Equation of line $F_+F_-\colon L = 0$.
Perpendicular to Euler line at $G$: $L_1 = 0$,
intersecting Kiepert hyperbola at $Y_1$.
Circle $F_+F_-G$ is one in the pencil of conics through $F_+, F_-, G$ and $Y_1$:
$$ (b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy - L \cdot L_1 = 0 $$
by suitably adjusting the linear forms $L, L_1$ by constants.
It center lies on the perpendicular bisector of $F_+F_-$. 
For arbitrary $t$, let $L_t = (1 - t)L_0 + t \cdot L_1$.
The line $L_t = 0$ is perpendicular to the Euler line.
The equation
\[(b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy - L \cdot L_t = 0\]
represents a circle through the Fermat points.

Figure 19. Gibert’s generalization of Lester’s circle
The line joining the midpoints of $HY_0$ and $GY_1$ contains the midpoint of the every chord cut out by $L_t = 0$. This line is also the perpendicular bisector of $F_+F_-$. Therefore the center of the circle is the midpoint of the chord.

Figure 20. Gibert’s generalization of Lester’s circle
3. The symmedian and isodynamic points

If $P, Q$ divide $X, Y$ harmonically, then $P$ and $Q$ are inverse in the circle with diameter $XY$. 

Figure 21
If $P$ and $Q$ are inverse in a circle $C$, then every circle through $P$ and $Q$ is orthogonal to $C$. 
Figure 23.
Consider three circles each orthogonal to the circumcircle and with center on a sideline of triangle $ABC$.

Their centers are collinear, and are on the pole of the **symmedian point** $K = (a^2 : b^2 : c^2)$.

They have two common points $J_+$ and $J_-$ called the **isodynamic points**, which are on the line $OK$ (**Brocard axis**), and are inverse in the circumcircle.

Every circle through $J_+$ and $J_-$ is orthogonal to the circumcircle.
The isodynamic points have coordinates

\[
J_+ = (a^2(\sqrt{3}S_A + S), b^2(\sqrt{3}S_B + S), c^2(\sqrt{3}S_C + S)),
\]

\[
= \sqrt{3}(a^2 S_A, b^2 S_B, c^2 S_C) + S(a^2, b^2, c^2);
\]

\[
J_- = \sqrt{3}(a^2 S_A, b^2 S_B, c^2 S_C) - S(a^2, b^2, c^2).
\]

They divide \(O\) and \(K\) harmonically.

Therefore, every circle through \(J_\pm\) is orthogonally to the **Brocard circle** (with diameter \(OK\)).

![Figure 24. The Brocard circle and the isodynamic points](image-url)
The isodynamic points are the only points whose \textbf{pedal triangles} are equilateral.

Figure 25. The pedal triangle of $J_+$ is equilateral
The isodynamic points are the **isogonal conjugates** of the Fermat points.

Figure 26. $J_+ = \text{isogonal conjugate of } F_+$
4. The first Evans circle

The **excentral triangle** $I_a I_b I_c$ has circumradius $2R$ and circumcenter $I' :=$ reflection of $I$ in $O$.

![The excentral triangle and its circumcircle](image-url)
The triangle of reflections

Figure 28. The triangle of reflections
The Evans perspector $W$ of the excentral triangle and the triangle of reflections

Figure 29. The Evans perspsector $W$
Let $I_aA^*$ intersect $OI$ at $W$. A routine calculation shows that

$$I'W : WI = R : -2r.$$ 

Similarly, $I_bB^*$ and $I_cC^*$ intersect $OI$ at points given by the same ratio. Therefore the lines $I_aA^*$, $I_bB^*$ and $I_cC^*$ concur at $W$ on $OI$. 

Figure 30. The Evans perspsector $W$ as a point on $OI$. 
Theorem 5. The Evans perspector $W$ and the incenter $I$ are inverse in the circumcircle of the excentral triangle.

Proof. $I'W \cdot I'I = \frac{R}{R-2r} \cdot I'I^2 = \frac{R^2}{R(R-2r)} \cdot (2 \cdot OI)^2 = (2R)^2$. \qed
Evans also found that the excentral triangle is perspective with each of the Kiepert triangles $\mathcal{K}\left(\frac{\pi}{3}\right)$ and $\mathcal{K}\left(-\frac{\pi}{3}\right)$. He denoted these perspectors by $V_+$ and $V_-$ and conjectured that $V_+, V_-, I$ and $W$ are concyclic.

Figure 32. Evans' perspector $V_-$ of $\mathcal{K}\left(-\frac{\pi}{3}\right)$ and excentral triangle
Proposition 6. Let $XBC$ and $X'I_bI_c$ be oppositely oriented similar isosceles triangles with bases $BC$ and $I_bI_c$ respectively. The lines $I_aX$ and $I_aX'$ are isogonal with respect to angle $I_a$ the excentral triangle.

Figure 33. Isogonal lines joining $I_a$ to apices of similar isosceles on $BC$ and $I_bI_c$
Proof. Triangles $XB I_a$ and $X' I_b I_a$ are similar since
\[ \angle XBI_a = \angle X'I_b I_a = \frac{\pi}{2} - \frac{B}{2} - \theta, \]
and as $BC$ and $I_b I_c$ are antiparallel,
\[ XB : X'I_b = BC : I_b I_c = I_a B : I_a I_b. \]
It follows that $\angle BI_a X = \angle I_b I_a X'$ and
the lines $I_a X$, $I_a X'$ are isogonal in the excentral triangle.
Theorem 7. Let $XYZ$ be the Kiepert triangle $\mathcal{K}(\theta)$ of $ABC$. The lines $I_aX$, $I_bY$, $I_cZ$ concur at a point $V(\theta)$ which is the isogonal conjugate of $K_e(-\theta)$ in the excentral triangle.

Proof. (i) $I_aX'$, $I_bY'$, $I_cZ'$ concur at the Kiepert perspector $K_e(-\theta)$ of the excentral triangle.

(ii) Since $I_aX$ and $I_aX'$ are isogonal with respect to $I_a$, and similarly for the pairs $I_bY$, $I_bY'$ and $I_cZ$ and $I_cZ'$, the lines $I_aX$, $I_bY$, $I_cZ$ concur at the isogonal conjugate of $K_e(-\theta)$ in the excentral triangle. □

Corollary 8. $V_\pm$ are the isodynamic points of the excentral triangle.

Therefore, every circle through $V_+$ and $V_-$ is orthogonal to the circumcircle of the excentral triangle.

If such a circle contains the incenter $I$, it also contains the inverse of $I$ in the circumcircle of the excentral triangle.

This latter is the Evans perspector $W$. 
**Theorem 9** (Evans). The points $V_{\pm}$ are concyclic with $I$ and $W$.

![Figure 35. The first Evans circle](image)

- $X_{1019}$
Proposition 10. The center of the first Evans circle is the point

\[ X_{1019} = \left( \frac{a(b-c)}{b+c}; \frac{b(c-a)}{c+a}; \frac{c(a-b)}{a+b} \right). \]

Figure 36. The first Evans circle
5. The Parry circle and the Parry point

The **Parry circle** is the one passing through the **isodynamic points** $J_{\pm}$ and the **centroid** $G$.

Since $J_{\pm}$ are inverse in the Brocard circle, the Parry circle is orthogonal to the Brocard circle, and also contains the inverse of $G$ in the Brocard circle.

The same is true with the Brocard circle replaced by the circumcircle.
Theorem 11. The inverse of the centroid $G$ in the Brocard circle is the Euler reflection point $E$.

Proof. The equation of the Brocard circle is
\[(a^2 + b^2 + c^2)(a^2yz + b^2zx + c^2xy) - (x+y+z)(b^2c^2x + c^2a^2y + a^2b^2z) = 0.\]

The polar of the centroid is the line
\[(b^2 - c^2)^2 x + (c^2 - a^2)^2 y + (a^2 - b^2)^2 z = 0.\]

This clearly contains the Euler reflection point
\[E = \left( \frac{a^2}{b^2 - c^2} : \frac{b^2}{c^2 - a^2} : \frac{c^2}{a^2 - b^2} \right),\]
which also lies on the line
\[\sum (b^2 - c^2)(S_{AA} - S_{BC})x = 0\]
joining $G$ to the midpoint of $OK$. $\square$
The lines $GE$ and $F_+F_-$ are parallel.
Since the Parry circle is orthogonal to the circumcircle, the polar $O$ is the radical axis of the circles. This line passes through the symmedian point $K$. The **Parry point** $P$ is the second intersection of the Parry circle and the circumcircle. It lies on a number of interesting circles.

![Figure 39. The Parry circle and Parry point](image)
(1) The circle $F_+ F_- G$ contains the Parry point $P$. 

Figure 40. The circle through $F_+ F_- G$ contains the Parry point
(2) **The circle** $OGK$ **contains the Parry point** $P$. 

![Diagram](image.png)

Figure 41. The circle $OGK$ contains the Parry point $P$
**Proposition 12.** The circle $F_+ F_- G$ intersects the circumcircle at the Parry point and the reflection of $E$ in the Euler line.
Proposition 13. The circle $OKG$ intersects the circumcircle at the Parry point $P$ and the reflection of $E$ in the Brocard axis.