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Linearization of families of germs of hyperbolic vector fields
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In this article, we develop some techniques to linearize families of smooth vector fields in a neighbourhood of a hyperbolic equilibrium point. In particular, we present the linearizing conjugacy in an explicit way and describe the smoothness of the conjugacy in terms of the eigenvalues of the vector fields.

\textbf{Keywords:} differential equations; normalization; linearization; resonance

\textbf{2000 Mathematics Subject Classifications:} Primary 34K17; 37C15

1. Introduction

Let $\mathcal{U}$ be an open neighbourhood of the origin in the parameter space $\mathbb{R}^d$, $d \in \mathbb{N}$, and $\mathcal{X}_\varepsilon$, $\varepsilon \in \mathcal{U}$, be a family of smooth vector fields defined in a vicinity of the origin which is assumed to be a hyperbolic equilibrium point $0 \in \mathbb{R}^N$. Write

$$\mathcal{X}_\varepsilon = \mathcal{X}_{l,\varepsilon} + \mathcal{X}_{h,\varepsilon},$$

where $\mathcal{X}_{l,\varepsilon}$ and $\mathcal{X}_{h,\varepsilon}$ represent, respectively, the linear part and the higher-order terms of $\mathcal{X}_\varepsilon$.

In terms of an individual vector field, i.e. for the fixed value of the parameter $\varepsilon = 0$, we know that there is an integer $k = k_0$, $0 \leq k \leq \infty$, such that $\mathcal{X}_0$ is $C^{k_0}$ conjugate to its linear approximation $\mathcal{X}_{l,0}$ [1]. We can even give an asymptotic expression of the linearizing conjugacy in an explicit way [2]. For families of vector fields, however, the situation is essentially different, since, in such a case, some resonances which do not occur at $\varepsilon = 0$ perhaps cannot be avoided at other values $\varepsilon$ close to 0. In fact, to our best knowledge, no answer to the following questions is known.

- Is it possible to linearize a family of vector fields $\mathcal{X}_\varepsilon$ via a $C^k$ change of coordinates $\Phi_\varepsilon$ for all $\varepsilon \in \mathcal{U}$, where the value of $k$ can be determined, in somehow simple way, say, in terms of the eigenvalues of $\mathcal{X}_{l,0}$?
- Is it possible to give, preferably in an explicit way, the expression of the above change of coordinates $\Phi_\varepsilon$?

Clearly, to have an affirmative answer to these questions is very interesting, not only from the differentiability point of view, but also from the point of view of existence and smoothness of invariant manifolds (such as the so-called extended

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unstable manifolds, see, for instance, [3]). It is particularly interesting in the case when we need to compute the Dulac map associated with $\mathcal{X}_e$, i.e. the Poincaré transition map associated with $\mathcal{X}_e$ from $S^-$ to $S^+$, where $S^-$ and $S^+$ are sections transverse to the stable and unstable manifolds of the vector field, respectively. This map plays an important role, for instance, in homoclinic bifurcation theory. For example, Roussarie [4] in this way succeeds to give an asymptotic expression of the Dulac map in $\mathbb{R}^2$ in terms of compensator functions (see Example 2 therein). In $\mathbb{R}^3$, a similar yet weaker version of the result is given in [5]. To be more precise, in the three-dimensional case, the authors show that the expression of the Dulac map can be expressed in terms of the Mourtada-type functions. See [5] and [6] for more details. So far, there is no general expression in higher dimensions.

The present article is devoted to giving positive answers to the aforementioned questions. To state the main result of this article, we need to introduce some notation and definitions. First of all, we shall denote the coordinates $z$, the unstable $x$ and stable coordinates $y$, respectively, by the symbols

$$z = (z_1, \ldots, z_q, z_{q+1}, \ldots, z_N) = (x, y)$$

$$= (x_1, \ldots, x_q, y_1, \ldots, y_p), \quad p + q = N.$$  \hspace{1cm} (1.2)

We denote the set of the eigenvalues of $\mathcal{X}_{t,e}$ by

$$\lambda(e) = (\lambda_1(e), \ldots, \lambda_q(e), \lambda_{q+1}(e), \ldots, \lambda_N(e))$$

$$= (\alpha(e), \beta(e))$$

$$= (\alpha_1(e), \ldots, \alpha_q(e), -\beta_1(e), \ldots, -\beta_p(e)),$$ \hspace{1cm} (1.3)

where $\alpha_j$ and $\beta_j$ are strictly positive numbers and are ordered in the following way:

$$0 < \alpha_1 < \cdots < \alpha_q; \quad 0 < \beta_1 < \cdots < \beta_p.$$ \hspace{1cm} (1.4)

Notice that all the eigenvalues in fact depend on the parameter $e$. However, as long as no confusion arises, we shall not specify the dependence.

**Definition 1.1:** Let $v \geq 0$ be a small real number and $e \in U$. An eigenvalue $\lambda_j(e)$ of $\mathcal{X}_{t,e}$ is said to admit a $v$-quasi-resonant condition if there are non-negative integer tuples $m = (m_1, \ldots, m_p)$ and $n = (n_1, \ldots, n_q)$, $|m| + |n| \geq 2$, where $|m| = m_1 + \cdots + m_p$, $|n| = n_1 + \cdots + n_q$, such that

$$\frac{|(\alpha(e), n) - (\beta(e), m) - \lambda_j(e)|}{|m| + |n| + 1} \leq v,$$ \hspace{1cm} (1.5)

where

$$(\alpha(e), n) = \alpha_1 n_1 + \cdots + \alpha_q n_q, \quad (\beta(e), m) = \beta_1 m_1 + \cdots + \beta_p m_p.$$ 

**Definition 1.2:** Given a non-linear term $z^\ell$ attached to $\partial^p \partial z_i$, i.e. $z^\ell(\partial^p \partial z_i) = x^m y^n (\partial^p \partial z_i)$, we define its order to be $|r| = |m| + |n|$, and we call the number

$$\delta_{e}(\lambda_j, m, n) = (\alpha(e), n) - (\beta(e), m) - \lambda_j(e),$$ \hspace{1cm} (1.6)
the eccentricity of the term. We call this term a \( \nu \)-quasi-resonant monomial if the following inequality stands:

\[
|\delta| \leq \nu(|m| + |n| - 1). \tag{1.7}
\]

It is clear that zero-quasi-resonance coincides with the classical definition of resonance \( \lambda_\epsilon(\epsilon) = (\alpha(\epsilon), n) - (\beta(\epsilon), m) \). Also, one can see that if \( \delta \) is of the \( \nu_1 \)-eccentricity of \( \lambda_\epsilon \), i.e. \( |\delta| \leq \nu_1(|m| + |n| - 1) \), then for any \( \nu_2 \geq \nu_1 \) we have \( |\delta| \leq \nu_2(|m| + |n| - 1) \). Therefore, it is of \( \nu_2 \)-eccentricity.

Throughout the article, we shall denote by

\[
v_0 = \sup_{\epsilon \in \mathbb{R}} \max_{1 \leq i \leq N} |\lambda_i(\epsilon) - \lambda_i(0)|. \tag{1.8}
\]

**Definition 1.3:** Let \( k \geq 0 \) be a given integer. We say that the eigenvalues of \( \mathcal{X}_{l,\epsilon} \) satisfy the \( P(k) \)-condition if for each \( \nu \)-quasi-resonant condition of the form (1.7), at least one of the following inequalities holds:

\[
k\alpha_\epsilon(\epsilon, n) = \alpha_1n_1 + \cdots + \alpha_qn_q, \tag{1.9}
\]

or

\[
k\beta_\epsilon(\epsilon, m) = \beta_1m_1 + \cdots + \beta_pm_p. \tag{1.10}
\]

**Example 1.4:** Let \( \lambda(\epsilon) = (1, 2 + \epsilon) \). Then for any given number \( \nu > 0 \), \( \alpha_3(\epsilon) \) admits a \( \nu \)-quasi-resonant relation when \( |\epsilon| < 3\nu \).

If \( \epsilon > 0 \), then \( \lambda(\epsilon) \) satisfies \( P(0) \)-condition; if \( \epsilon \leq 0 \), then \( \lambda(\epsilon) \) satisfies \( P(1) \)-condition.

**Theorem 1.5:** Let \( \mathcal{X}_\epsilon = \mathcal{X}_{l,\epsilon} + \mathcal{X}_{h,\epsilon} \) be a family of \( C^\infty \) vector field germs at a hyperbolic singular point \( 0 \in \mathbb{R}^p \), where \( \mathcal{X}_{l,\epsilon} \) is the linear part of \( \mathcal{X}_\epsilon \) and \( \mathcal{X}_{h,\epsilon} \) denotes the non-linear terms. Let \( \nu > 0 \) be a small real number. Assume that \( \mathcal{X}_{l,0} \) is semi-simple and its eigenvalues satisfy the \( P(k_0) \)-condition. Then there exist a neighbourhood \( \mathcal{W} \in \mathbb{R}^q \times \mathbb{R}^p \) at the origin and a \( C^{k_0} \) diffeomorphism \( \Phi_\epsilon : \mathcal{W} \to \mathbb{R}^q \times \mathbb{R}^p \) such that \( \Phi_\epsilon \) conjugates \( \mathcal{X}_{l,\epsilon} \) with \( \mathcal{X}_\epsilon \).

Notice that if \( \mathcal{X} \) is an individual germ of a vector field having eigenvalues \( \lambda_1, \ldots, \lambda_N \) satisfying the \( P(k) \) condition, then for sufficiently small parameter \( \gamma \) any deformation of \( \mathcal{X} \) is linearizable by a \( C^k \) local change of coordinates depending smoothly on the parameters \( \gamma \).

**Example 1.6:** Assume that \( \mathcal{X}_\epsilon \) is given as follows:

\[
\dot{x} = x + x^2y, \quad \dot{y} = (-1 + \epsilon)y.
\]

Then it is easy to see that, in this case, the eigenvalues satisfy \( P(1) \) condition. By Theorem 1.5, we know that there exists a \( C^1 \) diffeomorphism which can linearize the vector field. In fact, we can present this \( C^1 \) diffeomorphism in the following explicit way:

\[
x = \frac{\tilde{x}}{1 + \tilde{x}y \ln |\tilde{x}|}, \quad y = \tilde{y} \text{ if } \epsilon \neq 0
\]

\[
x = \frac{\tilde{x}}{1 - \tilde{x}y}, \quad y = \tilde{y} \text{ if } \epsilon = 0.
\]
It is straightforward to check that in the new coordinates \((\tilde{x}, \tilde{y})\), the vector field is linear:

\[
\begin{align*}
\dot{\tilde{x}} &= \tilde{x}, \\
\dot{\tilde{y}} &= (-1 + \varepsilon)\tilde{y}.
\end{align*}
\]

We point out that if \(k_0 = 0\) in Theorem 1.5, then the topological linearization follows from the classical result of the Hartman–Grobman theorem. Also, the \(C^{k_0}\) linearization in the theorem perhaps is not the best in the sense that the number \(k_0\) in the theorem is not possibly maximal. The price of such a possible loss is paid as the explicit expression of the linearizing diffeomorphism can be specified in Theorem 1.9.

**Definition 1.7:** Consider the following function \(f: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}\) having the form

\[
\begin{align*}
f(z, s) &= C(s) \exp \left( v(a_1|s_1| + \cdots + a_n|s_n|) \right) z^i \\
&= C(s_1, \ldots, s_n) \exp \left( v(a_1|s_1| + \cdots + a_n|s_n|) \right) z_{i_1} \cdots z_{i_m},
\end{align*}
\]

where the map \(s \mapsto C(s)\) together with their partial derivatives are smooth functions such that for each tuple \(j = (j_1, \ldots, j_n) \in \mathbb{N}^n\) there exists \(K_j > 0\) such that

\[
|\frac{\partial^{|j|} C(s)}{\partial s^j}| = |C^{(|j|)}(s)| \leq K_j.
\]

The number \(d_0(f) = \max_i |i_1 + \cdots + i_m|\) is called the order of \(f\) in \(z\).

The function \(f\) is called \(v\)-tagged monomial function if \(d_0(f) \geq |a_1| + \cdots + |a_n| + 1\).

In this article, we shall in fact mainly consider the cases where \(n = 1, 2\). In such cases, to avoid more subscripts, we shall explicitly denote \(s = s\) or \(s = (s, t)\), respectively.

**Definition 1.8:** A function \(f: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}\) is called a \(v\)-tagged function if there exists a sequence of \(v\)-tagged monomial functions \(\{h_i(z, s)\}_{i \in \mathbb{N}}\) such that

\[
f(z, s) = \sum_{i \in \mathbb{N}} h_i(z, s),
\]

and the order of \(f\) is defined by

\[
d_0(f) = \min_{i \in \mathbb{N}} d_0(h_i(z, s)).
\]

More generally, a vector function

\[
f: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^l(z, s), \quad f = (f_1(z, s), \ldots, f_l(z, s))
\]

is called a \(v\)-tagged function if each component \(f_i\) is a \(v\)-tagged function.

**Theorem 1.9:** Let \(\Phi_\varepsilon\) be the diffeomorphism given in Theorem 1.5 which conjugates \(X_{L,\varepsilon}\) with \(X_\varepsilon\). Then each component \(\Phi_i\) of \(\Phi_\varepsilon\), \(i = 1, \ldots, N\), is of the form

\[
\Phi_i(x, y) = \varphi_i \left( x, y, \log \left( \sum_{i=1}^{q} |x_i|^{\lambda_1 \cdots \lambda_N / \alpha_i} \right), \log \left( \sum_{i=1}^{p} |y_i|^{\lambda_1 \cdots \lambda_N / \beta_i} \right) \right),
\]

where \(\varphi_i\) is a \(v\)-tagged function.
In the special cases where all the eigenvalues of the vector fields have the same sign, say, positive ones, then the linearization conjugacy has a simpler form:

\[
\Phi_i(x) = \varphi_i \left( x, \log \left( \sum_{j=1}^{q} a_j |x_i|^{\alpha_j} \right) \right),
\]

where \( \varphi_i \) is a \( \nu \)-tagged function.

The article is organized as follows: In Section 2, we shall recall basic normal form theory. We shall also show in this section that, by the Poincaré–Dulac normal form procedure, \( X_{h,e} \) can be chosen to contain \( \nu \)-quasi-resonant terms only. Heavy notation and two propositions are collected in this section. The technical details of the proof are postponed to the Appendix. In Section 3, we shall mainly exhibit the normalization procedures so that the proof of the theorems given in Section 4 looks more compact. Since the estimation of the degree of differentiability of the conjugacy established in Section 3 is very technical, therefore we leave this part to the Appendix.

2. Preliminaries

In this part of the article, we shall introduce more notation and definitions. We denote by

\[
\rho_{\text{sup}} = \sup_{\varepsilon \in L} \max_i |\lambda_i|
\]
\[
= \sup_{\varepsilon \in L} \max_i \{\alpha_\varepsilon, \beta_\varepsilon\}, \tag{2.1}
\]

\[
\rho_{\text{inf}} = \inf_{\varepsilon \in L} \min_i |\lambda_i|
\]
\[
= \inf_{\varepsilon \in L} \min_i \{\alpha_\varepsilon, \beta_\varepsilon\}. \tag{2.2}
\]

Recall the notation (1.2) and (1.3). If \( x^a y^m \partial/\partial z_i \) is a \( \nu \)-quasi-resonant monomial, by definition we have

\[
(\alpha_\varepsilon, n) - \delta \cdot \left( x^a y^m \partial/\partial z_i \right) = (\beta_\varepsilon, m) + \lambda_\varepsilon. \tag{2.3}
\]

From the relation (2.3), we have

\[
\rho_{\text{inf}} |n| - \nu(|r| + 1) \leq \rho_{\text{sup}}(|m| + 1).
\]

By adding \( \rho_{\text{inf}} |m| \) on the both sides of the last inequality, we obtain

\[
\rho_{\text{inf}} |r| - \nu(|r| + 1) \leq (\rho_{\text{sup}} + \rho_{\text{inf}})(|m| + 1). \tag{2.4}
\]

Similarly, we have

\[
\rho_{\text{inf}} |r| - \nu(|r| + 1) \leq (\rho_{\text{inf}} + \rho_{\text{sup}})(|n| + 1). \tag{2.5}
\]
Now from (2.4) and (2.5), we have the following:

**Proposition 2.1:** If \( z^r \partial / \partial z_i = x^m \partial / \partial z_i \) is a \( v \)-quasi-resonant monomial, then the following properties hold:

1. \( n \) \( m \) \( K_1 |n| \), \( n \) \( K_2 |m| \).

2. If \( |n| \neq 0 \) and \( |m| \neq 0 \), then there exist \( K_1 > 0 \) and \( K_2 > 0 \) such that

\[
|m| \leq K_1 |n|, \quad |n| \leq K_2 |m|.
\]

3. If a non-linear term is a \( v \)-quasi-resonant term of \( \mathcal{X}_{1,0} \), then it is a \( (v + v_0) \)-quasi-resonant term of \( \mathcal{X}_{1,e} \).

4. If a non-linear term is not a \( v \)-quasi-resonant term of \( \mathcal{X}_{1,0} \), then it is not a \( (v - v_0) \)-quasi-resonant term of \( \mathcal{X}_{1,e} \).

In fact, the properties [1-3] and [1-4] are obvious. To see the point [1-1], notice that if \( |m| = 0 \), then from (2.4) we have the inequality

\[
|n| \leq \frac{\rho_{\text{sup}} + \rho_{\text{inf}}}{\rho_{\text{inf}} - v}.
\]

By the same argument, if \( |n| = 0 \), then from (2.5) we have the inequality

\[
|m| \leq \frac{\rho_{\text{sup}} + \rho_{\text{inf}}}{\rho_{\text{inf}} - v}.
\]

To see the point [1-2], from (2.4) or (2.5) we obtain

\[
(\rho_{\text{inf}} - 2v)|r| \leq 2(\rho_{\text{inf}} + \rho_{\text{sup}})|m|
\]

and

\[
(\rho_{\text{inf}} - 2v)|r| \leq 2(\rho_{\text{inf}} + \rho_{\text{sup}})|n|.
\]

This means that for \( v \) sufficiently small, there exist \( K_1 > 0 \) and \( K_2 > 0 \) such that

\[
|r| \leq K_1 |n|, \quad |r| \leq K_2 |m|.
\]

Below we introduce the convolution of two functions \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) as follows.

\[
f * g(t) = \int_0^t f(t - s)g(s)ds.
\]

Throughout the article, we shall fix the following functions

\[
\mathcal{E}(s) = \exp(\varepsilon s), \quad \chi = \mathcal{E}_0(s) \equiv 1.
\]

**Example 2.1:** The following identities hold.

\[
\mathcal{E}_1 * \mathcal{E}_2(t) = \frac{1}{\varepsilon_2 - \varepsilon_1} \left( e^{\varepsilon_2 t} - e^{\varepsilon_1 t} \right) \quad \text{if} \quad \varepsilon_1 \neq \varepsilon_2.
\]

\[
\mathcal{E}_e * \mathcal{E}_e(t) = e^{\varepsilon t}I.
\]

\[
\chi * \mathcal{E}_e(t) = \frac{1}{\varepsilon}(e^{\varepsilon t} - 1);
\]

\[
(\chi * \mathcal{E}_e(t))' = \varepsilon \chi * \mathcal{E}_e(t) + 1.
\]
In [4], the function \( x \mapsto \chi * \mathbb{T}_\varepsilon(\log |x|) = ((|x|^\varepsilon - 1)/\varepsilon) \) is called a compensator. We refer the reader to [6] for more details about the compensator functions in asymptotic analysis of the Dulac maps.

The proof of the following proposition is purely technical and will be given in the Appendix.

**Proposition 2.2:** A smooth function \( F : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R} \) of the following form is a 2\( v \)-tagged function

\[
F(z, s) = z^l \chi * \mathbb{T}_{\mu_1} \cdots * \mathbb{T}_{\mu_l}(s)
\]

where \( |i| \geq l + 1 \), and \( \mu_1, \ldots, \mu_l \) are some real numbers satisfying the relations \( \mu_j \leq v_j \).

We now rewrite \( \mathcal{X}_e \) given by (1.1) into the form

\[
\mathcal{X}_e = \mathcal{X}_{L,e} + \mathcal{X}_{R,e} + \mathcal{X}_{S,e} + \mathcal{X}_{f,e},
\]

(2.9)

where \( \mathcal{X}_{R,e} \) consists of \( v \)-quasi-resonant monomials of \( \mathcal{X}_{L,0} \), \( \mathcal{X}_{S,e} \) consists of the rest monomials, namely, those monomials which are not \( v \)-quasi-resonant of \( \mathcal{X}_{L,0} \) (therefore, they are not \( v/2 \)-quasi-resonant of \( \mathcal{X}_{L,e} \)), and \( \mathcal{X}_{f,e} \) consists of flat terms. By the Poincaré-Dulac resonant normal form, we can find a smooth change of variables such that \( \mathcal{X}_e \) is formally conjugate with \( \mathcal{X}_{L,e} + \mathcal{X}_{R,e} \). Moreover, according to the Chen theorem [7], the conjugacy can even be chosen to be smooth. Thus, to study \( \mathcal{X}_e \), it suffices to consider (2.9) where the term \( \mathcal{X}_{S,e} \) is deleted, i.e.

\[
\mathcal{X}_e = \mathcal{X}_{L,e} + \mathcal{X}_{R,e}.
\]

(2.10)

We also put the proof of the following proposition in the Appendix.

**Proposition 2.3:** Let \( \tilde{f}, \tilde{g} : \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \) be some smooth functions having the form

\[
\tilde{f}(z, s, t) = \left( f(z, s, t), s, t \right), \quad \tilde{g}(z, s, t) = \left( g(z, s, t), s, t \right),
\]

where \( f \) and \( g \) are \( v \)-tagged functions. Then the following properties hold:

[3-1] For any \( \mu, \lambda \in \mathbb{R} \), the map \( (z, s, t) \mapsto \lambda f(z, s, t) + \mu g(z, s, t) \) is a \( v \)-tagged function.

[3-2] The map \( (z, s, t) \mapsto f(z, s, t)g(z, s, t) \) is a \( v \)-tagged function, where \( f \cdot g = (f_1g_1, \ldots, f_mg_m) \).

[3-3] The map \( (z, s, t) \mapsto f(g(z, s, t)) \) is a \( v \)-tagged function.

[3-4] Assume that

\[
f(z, s, t) = z + H(z, s, t)
\]

(2.11)

where \( H(z, s, t) = (H_1(z, s, t), \ldots, H_N(z, s, t)) \) and

\[
H_i(z, s, t) = \sum_{|k|=1}^{\infty} C_i^{[k]}(s, t)z^{M_i^{[k]}} \exp \left( v \left( w_{i,1}^{[k]} s + w_{i,2}^{[k]} t \right) \right),
\]

(2.12)
Then \( \tilde{f} \) is invertible, \( \tilde{f}^{-1}(z, s, t) = (z + h(z, s, t), s, t) \), and \( h(z, s, t) \) is a \( \nu \)-tagged function of the form \( h = (h_1, \ldots, h_N) \), where

\[
h_i(z, s, t) = \sum_{|k|=1}^{\infty} C_i^{(k)}(s, r) z_i^{M_i^{(k)}} \exp\left(\nu \left(w_i^{(k)}|s| + w_i^{(k)}|t|\right)\right)
\]

(2.13)

Moreover, the following properties hold:

[3-4-1] If \( z_i^{M_i^{(k)}} \) in (2.12) is \( \nu \)-quasi-resonant with respect to \( \mathcal{X}_{i \ell} \), then so is \( z_i^{M_i^{(k)}} \) in (2.13).

[3-4-2] If \( z_i^{M_i^{(k)}} \) in (2.12) satisfies the \( P(k_0) \) condition, then \( z_i^{M_i^{(k)}} \) in (2.13) satisfies the \( P(k_0) \) condition too.

In the next section, we develop a normalization procedure to eliminate those \( \nu \)-quasi-resonant monomials.

### 3. Normalization procedures and proof of the theorem

Consider the following germ of vector field

\[
\hat{\mathcal{X}}_\epsilon = \mathcal{X}_\epsilon + \frac{\partial}{\partial t} + \frac{\partial}{\partial s}.
\]

(3.1)

In other words, we introduce two extra variables, \( t \) and \( s \), which shall be called tags of the vector field \( \mathcal{X}_\epsilon \). We assume that the tags satisfy

\[
i = \dot{s} = 1,
\]

where the derivatives are taken with respect to \( \tau \), the time variable of the original vector field \( \mathcal{X}_\epsilon \). We remark that the variables \( s \) and \( t \) will depend on the final coordinate system in which the germ represented by Equation (3.2) is linearized (see below).

For a given monomial \( M_i = a x^\alpha y^\beta \partial / \partial z_i \), we define its tag \( \text{tag}(M_i) \) as follows.

\[
\text{tag}(M_i) = \begin{cases} 
  t & \text{if } \frac{(\alpha, \beta)}{\alpha_q} < \frac{(\beta, m)}{\beta_p} \\
  s & \text{if } \frac{(\alpha, \beta)}{\alpha_q} \geq \frac{(\beta, m)}{\beta_p}
\end{cases}
\]

Now consider the following ordinary differential equations associated with the vector field (3.1).

\[
\begin{cases}
  \dot{z}_i = \lambda_i z_i + \sum_{|k|=2}^{\infty} M_i^{k,1}(z), & 1 \leq i \leq N, \\
  i = 1, \\
  \dot{s} = 1,
\end{cases}
\]

(3.2)

where all the terms \( M_i^{k,1}(z) \) are \( \nu \)-quasi-resonant monomials of \( \mathcal{X}_{i0} \) with the order \( |k| \).

The reason why we employ double superscripts will be clear shortly.
By the definition of \( \nu \)-quasi-resonant monomial, we know that there exist a sequence of eccentricities \( \{\delta_i^{k_i}\}_{k_i \in \mathbb{N}^\nu} \) such that
\[
\delta_i^{k_i} = |\delta_i(M_i^{k_i})| \leq \nu(d_0(M_i^{k_i}) + 1). \tag{3.3}
\]
Now by performing some straight calculation, one can obtain the following relation
\[
\mathcal{X}_i(M_i^{k_i}) = (\lambda_i - \delta_i^{k_i})M_i^{k_i} + M_i^{k_i-2}, \tag{3.4}
\]
where, due to the assumption that \( \mathcal{X}_i \) is semi-simple, one can see that the order of the terms in \( M_i^{k_i-2} \) is higher than \( |k| \), the order of \( M_i^{k_i} \). In other words, we have
\[
d_0(M_i^{k_i-2}) \geq d_0(M_i^{k_i}) + 1. \tag{3.5}
\]

Now the superscript digit 2 in \( M_i^{k_i-2} \) plays at least two roles: on the one hand, it carries the meaning that the terms \( M_i^{k_i-2} \) are obtained by the action of the vector field \( \mathcal{X}_i \) on \( M_i^{k_i} \); on the other hand, the inequality (3.5) describes the increasing order. Moreover, with such kind of superscript, we can make the process continue.

We have the following lemma.

**Lemma 3.1:** Let \( M_i^{k_i} \) and \( M_i^{k_i-2} \) be any two terms in (3.4) having the form
\[
M_i^{k_i} = x^m y^n, \quad M_i^{k_i-2} = \bar{x}^\bar{m} \bar{y}^\bar{n},
\]
then \( M_i^{k_i-2} \) is a \( \nu \)-quasi-resonant monomial and
\[
(\alpha, \bar{m}) \leq (\alpha, \bar{m}), \tag{3.6}
\]
and
\[
(\beta, n) \leq (\beta, \bar{n}). \tag{3.7}
\]

**Proof:** Denote by \( (m, n) = r = (r_1, \ldots, r_N) \). Thus, \( M_i^{k_i-1} = x^m y^n = z^r \). From (3.2) we have
\[
\mathcal{X}_i(M_i^{k_i-1}) = \sum_{j=1}^{N} r_j M_i^{k_i-1} z_j^{-1}\left(\lambda_j z_j + \sum_{|ll| \geq 2} k_j^{(|l|)} z^{M_i^{(|ll|)}}\right)
= (\lambda, r)M_i^{k_i-1} + \sum_{j=1}^{N} \sum_{|ll| \geq 2} r_j k_j^{(|l|)} M_i^{k_i-1} z_j^{-1} z^{M_i^{(|ll|)}}.
\]

Therefore up to a coefficient, we know that \( M_i^{k_i-2} \) has the form \( M_i^{k_i-1} z_j^{-1} z^{M_i^{(|ll|)}} \). It follows that if \( z^{M_i^{(|ll|)}} = x^m y^n \), then
\[
d_0(M_i^{k_i-2}) = |m| + |n| + |\bar{m}| + |\bar{n}| - 1 \tag{3.8}
\]
and
\[
(\alpha, \bar{m}) = (\alpha, m) + (\alpha, m) - \alpha_j. \tag{3.9}
\]

Since \( z^{M_i^{(|ll|)}} \) is a \( \nu \)-quasi-resonant monomial, therefore
\[
(\beta, \bar{n}) + (\alpha, \bar{m}) = \alpha_j - \delta_j
\]
where $|\delta_j| \leq \nu(1 + |	ilde{m}| + |	ilde{n}|)$. Notice that the tuples $\tilde{m}$ and $\tilde{n}$ depend only on the vector field. Therefore if $\nu > 0$ is chosen sufficiently small, then from (3.9) we have the following:

$$(\alpha, \tilde{m}) = (\alpha, m) + (\alpha, \tilde{m}) - \alpha_j$$

$$\geq (\alpha, m) + (\beta, \tilde{n}) - \nu(1 + |	ilde{m}| + |	ilde{n}|) (3.10)$$

$$\geq (\alpha, m).$$

The inequality (3.7) can be shown in the same way. Moreover, we see that the eccentricity of $M_{i}^{k, 2}$ satisfies

$$\delta_{e}(M_{i}^{k, 2}) \leq \nu_1(m_1 + \tilde{n}_1) + \cdots + \nu_q(m_q + \tilde{n}_q)$$

$$+ \beta_1(n_1 + \tilde{n}_1) + \cdots + \beta_p(n_p + \tilde{n}_p) - 2\nu_j$$

$$\leq \nu(|m| + |\tilde{n}| + |	ilde{m}|)$$

By (3.8), we know that $M_{i}^{k, 2}$ is a $\nu$-quasi-resonant monomial. Therefore, there exists a sequence of eccentrics of $M_{i}^{k, 2}$ such that for each integer tuple $k$, $|k| \geq 3$, $1 \leq i \leq N$ (cf (3.3))

$$\delta_{i}^{k, 2} = |\delta_{e}(M_{i}^{k, 2})| \leq \nu(d_0(M_{i}^{k, 2}) + 1).$$

3.1. The initial step of normalization

For each integer tuple $k \in \mathbb{N}^N$ from the non-linear term $M_{i}^{k, 1}$, we put $t_{i,k} = \text{tag}(M_{i}^{k, 1})$. Now consider the following change of variables:

$$z_{i}^{[1]} = z_{i} - \sum_{|k| = 2}^{\infty} M_{i}^{k, 1}(z) * \hat{g}_{i}^{k, 1}(t_{i,k}), \quad i = 1, \ldots, N. \tag{3.12}$$

It is easy to see that

$$z_{i}^{[1]} = \lambda_{i} z_{i} + \sum_{|k| = 2}^{\infty} M_{i}^{k, 1}(z) - \sum_{|k| = 2}^{\infty} \hat{\lambda}_{i}^{k}(M_{i}^{k, 1}(z)) \times \hat{g}_{i}^{k, 1}(t_{i,k})$$

$$- \sum_{|k| = 2}^{\infty} M_{i}^{k, 1} \hat{\lambda}_{i}^{k}(z * \hat{g}_{i}^{k, 1}(t_{i,k})). \tag{3.13}$$

By (3.4) and (2.8), we have

$$z_{i}^{[1]} = \lambda_{i} z_{i} + \sum_{|k| = 2}^{\infty} M_{i}^{k, 1}(z)$$

$$- \sum_{|k| = 2}^{\infty} (\alpha_{i} - \delta_{i}^{k, 1}) M_{i}^{k, 1} - \sum_{|k| = 2}^{\infty} \hat{\lambda}_{i}^{k}(M_{i}^{k, 1}(z)) \times \hat{g}_{i}^{k, 1}(t_{i,k})$$

$$- \sum_{|k| = 2}^{\infty} \delta_{i}^{k, 1} M_{i}^{k, 1}(z * \hat{g}_{i}^{k, 1}(t_{i,k})) - \sum_{|k| = 2}^{\infty} M_{i}^{k, 1}(z)$$

$$= \lambda_{i} z_{i}^{[1]} + \sum_{|k| = 2}^{\infty} M_{i}^{k, 2}(z) * \hat{g}_{i}^{k, 1}(t_{i,k}). \tag{3.14}$$
Notice that the monomial terms $M_i^{k,1}$'s disappear whereas the higher-order terms $M_i^{k,2}(z)\chi \neq \xi_{j_1}$ are created.

From Proposition 2.2, we know that both maps

$$(z, s, t) \mapsto M_i^{k,2}(z)\chi \neq \xi_{j_1}(t),$$

and

$$(z, s, t) \mapsto M_i^{k,2}(z)\chi \neq \xi_{j_1}(s),$$

are $2\nu$-tagged monomial functions.

### 3.2. The induction

Now we perform induction on changes of variables. Suppose that for an integer $r \geq 1$, the following equation holds.

$$z_i^{[r]} = \lambda_i z_i^{[r]} + \sum_{|k|=2}^\infty M_i^{k,r+1}(z)\varphi_{i,r,k}(t_{i,k}), \quad 1 \leq i \leq N,$$

(3.15)

where the function $\varphi$ is of the form

$$\varphi_{i,r,k}(t_{i,k}) = \chi \neq \xi_{j_1} \star \xi_{j_2} \star \cdots \star \xi_{j_r}(t_{i,k}).$$

Here $M_i^{k,r+1}(z)$'s are $\nu$-quasi-resonant monomials of $X_{t,0}$ with the orders satisfying (cf (3.5))

$$d_0(M_i^{k,r+1}) > d_0(M_i^{k,1}) + r \geq 2 + r.$$

Assume that there exists a sequence $\{\delta_i^{k,r+1}\}_{k \in \mathbb{N}^\nu}$ such that

$$|\delta_i^{k,r+1}| \leq \nu \left(d_0(M_i^{k,r+1}) + 1\right)$$

and

$$X(z_i^{k,r+1}(z)) = (\lambda_i - \delta_i^{k,r+1})M_i^{k,r+1}(z) - M_i^{k,r+2}(z),$$

(3.16)

where generally $M_i^{k,r+2}(z)$ is a collective term containing a set of all $\nu$-quasi-resonant monomials of $X_{t,0}$ attached to coordinate $z_i$. See (3.4) for a comparison.

Now perform the following inductive change of coordinates

$$z_i^{[r+1]} = z_i^{[r]} - \sum_{|k|=2}^\infty M_i^{k,r+1}(z)\varphi_{i,r,k} \neq \xi_{j_1}(t_{i,k}).$$

Then we have

$$z_i^{[r+1]} = \lambda_i z_i^{[r]} + \sum_{|k|=2}^\infty M_i^{k,r+1}(z)\varphi_{i,r,k}(t_{i,k}) - \sum_{|k|=2}^\infty M_i^{k,r+1}(z)\varphi_{i,r,k} \neq \xi_{j_1}(t_{i,k})$$

$$- \sum_{|k|=2}^\infty (\lambda_i - \delta_i^{k,r+1})M_i^{k,r+1}(z)\varphi_{i,r,k} \neq \xi_{j_1}(t_{i,k}) + M_i^{k,r+2}(z)\varphi_{i,r,k} \neq \xi_{j_1}(t_{i,k}).$$
By the relations (2.8), (3.1) and (3.4), we have

\[ z_i^{[r+1]} = \lambda_i z_i^{[r]} + \sum_{|k|=2}^{\infty} M_i^{k,r+1}(z) \psi_{i,r,k}(t_{i,k}) \]

\[ - \sum_{|k|=2}^{\infty} (\lambda_i - \delta_i^{k,r+1}) M_i^{k,r+1}(z) \psi_{i,r,k} \star \mathbb{T}^r_{\delta_i^{k,r+1}}(t_{i,k}) \]

\[ + M_i^{k,r+2}(z) \psi_{i,r,k} \star \mathbb{T}^r_{\delta_i^{k,r+1}}(t_{i,k}) \]

\[ - \sum_{|k|=2}^{\infty} M_i^{k,r+1}(z) \psi_{i,r,k}(t_{i,k}) \]

\[ - \sum_{|k|=2}^{\infty} \delta_i^{k,r+1} M_i^{k,r+1}(z) \psi_{i,r,k} \star \mathbb{T}^r_{\delta_i^{k,r+1}}(t_{i,k}) \]

\[ = \lambda_i z_i^{[r+1]} + \sum_{|k|=2}^{\infty} M_i^{k,r+2}(z) \varphi_{i,r+1,k}(t_{i,k}), \]

where

\[ \varphi_{i,r+1,k}(t_{i,k}) = \psi_{i,r,k} \star \mathbb{T}^r_{\delta_i^{k,r+1}}(t_{i,k}). \]

Since \( d_0(M_i^{k,r+2}) > d_0(M_i^{k,r+1}) \), it therefore turns out that either

\[ (z,s) \mapsto M_i^{k,r+2}(z) \varphi_{i,r+1,k}(s) \]

or

\[ (z,t) \mapsto M_i^{k,r+2}(z) \varphi_{i,r+1,k}(t) \]

is a \( v \)-tagged monomial function. Its degree in \( s \) or in \( t \) is \( r \), which satisfies the inequality \( d_0(M_i^{k,r+2}) > r + 3 \).

Observe that, up to coefficients, if

\[ M_i^{k,r}(z) = x^m y^n \]

and

\[ M_i^{k,r+1}(z) = x^{\tilde{m}} y^{\tilde{n}}, \]

then by Lemma 3.1, we have

\[ (\alpha, m) \leq (\alpha, \tilde{m}) \]

and

\[ (\beta, n) \leq (\beta, \tilde{n}). \]

Now, the \( P(k_0) \)-condition means the following facts: If \( t_{i,k} = t \), then

\[ k_0 \beta_p < (\beta, n) \leq (\beta, \tilde{n}), \]

(3.17)
and if \( t_{i,k} = s \), then
\[
k_{0}a_{i} < (\alpha, \mathbf{m}) \leq (\alpha, \mathbf{m}).
\] (3.18)

Following the property [1-2] in Proposition 2.1, we have the following inequalities
\[
|\mathbf{n}| + |\mathbf{m}| \leq K_{1}|\mathbf{n}|
\]
\[
|\mathbf{n}| + |\mathbf{m}| \leq K_{2}|\mathbf{m}|.
\] (3.19)

Therefore,
\[
r + 1 \leq d_{0}(M^{k,r}_{i}) \leq \min\{E|\mathbf{n}|, E|\mathbf{m}|\},
\] (3.20)
where \( K = \max(K_{1}, K_{2}) \).

4. A tagged transformation

From the above discussion and from Proposition 2.2, we know that for each integer \( r \in \mathbb{N} \) the following map is a 2\( \nu \)-tagged function.
\[
z^{[r]}_{i} = \Phi^{[r]}_{i}(\mathbf{z}) = z_{i} - \sum_{m=1}^{r} \sum_{|k|=2}^{\infty} H^{[k],m}_{i}(\mathbf{z}, s, t), \quad i = 1, \ldots, N,
\] (4.1)

where
\[
H^{[k],m}_{i}(\mathbf{z}, s, t) = M^{[k],m}_{i}(\mathbf{z})\phi_{i,m,k}(t_{i,k}).
\]

According to the above discussion, we know that \( d_{0}(H^{[k],m}_{i}) \) tends to \(+\infty\) as \( m \) tends to \(+\infty\).

Now in the new system, we have the following equation.
\[
\dot{z}^{[r]}_{i} = \lambda_{i}z^{[r]}_{i} + \sum_{|k|=2}^{\infty} H_{i}^{[k],r+1}(\mathbf{z}, s, t).
\] (4.2)

According to [3-4] in the Proposition 2.3 and the relation (4.1), we can express \( \mathbf{z} \) in terms of \( \mathbf{z}^{[r]} \). Thus, (4.2) has the following form
\[
\dot{z}^{[r]}_{i} = \lambda_{i}z^{[r]}_{i} + \sum_{|k|=2}^{\infty} \hat{H}_{i}^{[k],r+1}(\mathbf{z}^{[r]}, s, t),
\] (4.3)

where for each integer \( i \in \{1, \ldots, N\} \), \( d_{0}(\hat{H}^{[k],r+1}_{i}) \) tends to \(+\infty\) as \( r \) tends to \(+\infty\).

Let \( \Psi^{(r)} = (\Phi^{(r)}_{i})^{-1} \). This therefore implies that
\[
\left( (\Psi^{(r)}_{i}X_{i,s})(\mathbf{z}^{[r]}) \right)_{s} = \left( \mathcal{X}_{i}(\mathbf{z}^{[r]}) \right)_{s} + G_{i}^{[r]}(\mathbf{z}^{[r]}, s, t) \frac{\partial}{\partial z^{[r]}_{i}}, \quad i = 1, \ldots, N.
\] (4.4)

where
\[
\lim_{r \to \infty} d_{0}(G_{i}^{[r]}(\mathbf{z}^{[r]}, s, t)) = \infty.
\]
We now put

\[ s = s(x^{[r]}) = \log \left( \sum_{j=1}^{q} |x_j^{[r]}|^b \right), \]

\[ t = t(y^{[r]}) = -\log \left( \sum_{j=1}^{p} |y_j^{[r]}|^c \right), \tag{4.5} \]

where \( b_j = \lambda_1 \cdots \lambda_N/\alpha_j \) and \( c_j = \lambda_1 \cdots \lambda_N/\beta_j \).

Now for any fixed \( i \in \{1, \ldots, N\} \), we analyse the map

\[ z \mapsto G_i^{[r]} \left( z^{[r]}, s(x^{[r]}), t(y^{[r]}) \right). \tag{4.6} \]

First of all, since \( G_i^{[r]} \) is a 2\( \nu \)-tagged function, it takes the following form (where we drop all the superscripts \( \{r\} \) in the coordinates).

\[ G_i^{[r]}(z, s(x), t(y)) = \sum_{|k|=1}^\infty C_i^{[k]} \left( s(x), t(y) \right) z^{M_i^{[k]}} \cdot \left( \sum_{j=1}^{q} |x_j|^b \right)^{-2\nu w_i^{[k]}} \cdot \left( \sum_{j=1}^{p} |y_j|^c \right)^{-2\nu w_i^{[k]}}, \]

where for each integer tuple \( k \), \( C_i^{[k]}(s(x), t(y)) \) is a smooth function with bounded derivatives, \( M_i^{[k]} = (M_{i,1}^{[k]}, \ldots, M_{i,N}^{[k]}) \) and

\[ M_{i,1}^{[k]} + \cdots + M_{i,N}^{[k]} \geq w_{i,1}^{[k]} + w_{i,2}^{[k]} + 1. \]

Moreover, since \( z^{M_i^{[k]}} \) is a \( \nu \)-quasi-resonant monomial, by the properties in Proposition 2.1 we have

\[ E_1 \left( M_{i,1}^{[k]} + \cdots + M_{i,q}^{[k]} \right) \geq w_{i,1}^{[k]} + w_{i,2}^{[k]} + 1 > w_{i,1}^{[k]}, \]

\[ E_2 \left( M_{i,q+1}^{[k]} + \cdots + M_{i,N}^{[k]} \right) \geq w_{i,1}^{[k]} + w_{i,2}^{[k]} + 1 > w_{i,2}^{[k]}. \tag{4.7} \]

To proceed our discussion, we need the following proposition. Its proof involves certain technical details and is given in the Appendix.

**Proposition 4.1:** Let

\[ X = \sum_{j=1}^{q} |x_j|^b, \quad Y = \sum_{j=1}^{p} |y_j|^c, \]

where \( b_j = \lambda_1 \cdots \lambda_N/\alpha_j \) and \( c_j = \lambda_1 \cdots \lambda_N/\beta_j \). Let

\[ \psi_+(z, s) = \varphi^+(s)x^m \exp(2\nu w_1 |s|), \tag{4.8} \]

\[ \psi_-(z, t) = \varphi^-(t)y^m \exp(2\nu w_2 |t|), \tag{4.9} \]
where both \( \varphi^+(s) \) and \( \varphi^-(t) \) are smooth functions with bounded successive derivatives. Then the maps

\[
\mathcal{J}^+(z) = \psi_+(z, \log X), \quad \mathcal{J}^-(z) = \psi_-(z, \log Y)
\]

satisfy the following inequalities, respectively.

\[
\left| \frac{\partial^{[i]} \mathcal{J}^+(z)}{\partial x^i} \right| \leq O \left( |X|^{k_1} \right), \tag{4.10}
\]

\[
\left| \frac{\partial^{[i]} \mathcal{J}^-(z)}{\partial y^i} \right| \leq O \left( |Y|^{k_2} \right), \tag{4.11}
\]

where

\[
k_1 > (\alpha, m) - |j|\alpha_q - 2vw_1, \tag{4.12}
\]

\[
k_2 > (\beta, n) - |j|\beta_p - 2vw_2. \tag{4.13}
\]

Now applying this proposition to the function \( G_1^{[r]} \), we immediately have

\[
\left| \frac{\partial^{[i]} G_1^{[r]}(z^{[r]})}{\partial (x^{[r]})} \right| \leq O(|X|^{k_1^{(r)}}),
\]

\[
\left| \frac{\partial^{[i]} G_1^{[r]}(z^{[r]})}{\partial (y^{[r]})} \right| \leq O(|Y|^{k_2^{(r)}}),
\]

where

\[
k_1^{(r)} > \sum_{i=1}^{q} \alpha_i M_{i,j} - |j|\alpha_q - 2vw_{i,1},
\]

\[
k_2^{(r)} > \sum_{i=1}^{p} \beta_i M_{i,q+l} - |j|\beta_p - 2vw_{i,2}.
\]

Thus, by (4.7) we have

\[
k_1^{(r)} > \sum_{i=1}^{q} \alpha_i M_{i,j} - |j|\alpha_q - 2vK_1 \sum_{i=1}^{q} \alpha_i M_{i,j},
\]

\[
k_2^{(r)} > \sum_{i=1}^{p} \beta_i M_{i,q+l} - |j|\beta_p - 2vK_2 \sum_{i=1}^{p} \beta_i M_{i,q+l}.
\]

This implies that for any \( j \), as the induction goes on, \( r \) increases, and by (3.20), \( G_1^{[r]} \) is \( C^{[r]} \)-flat with respect to \( x \) or \( y \) at 0. Now letting \( r \to \infty \), we obtain (cf (4.4)) the following formal equality.

\[
(\Psi^\infty_* \mathcal{X}_{1,s}(z^{(\infty)}) = \mathcal{X}_t(z^{(\infty)}) + G^{(\infty)}(z^{(\infty)}, s, t) \cdot \frac{\partial}{\partial z^{(\infty)}},
\]

where \( G^{(\infty)} \) is a flat function which can be removed by the Chen theorem.
Denote by $\Psi = \Psi^{(\infty)}$. It remains to show that the map
$$z \mapsto \Psi(z, s(z), t(z)) = (\Psi_1, \ldots, \Psi_N)$$
is $C^{k_0}$, where $k_0$ is from the $P(k_0)$ condition given in the theorem. The expression written in (4.1) is a $2\nu$-tagged diffeomorphism. By the property [3-4] in Proposition 2.3, for each integer $i=1,\ldots,N$, $\Psi(z)$ has the form
$$\Psi_i(z) = z_i + \sum_{|k|=1}^\infty C_i^k(s, t)z^M_i \exp\left(2\nu\left(w_{i,1}^k|s| + w_{i,2}^k|t|\right)\right),$$
where for each $k$ the map
$$(z, s, t) \mapsto C_i^k(s, t)z^M_i \exp\left(2\nu\left(w_{i,1}^k|s| + w_{i,2}^k|t|\right)\right)$$
is a $2\nu$-tagged monomial function.

Fix the tuple $k \in \mathbb{N}^N$ and write
$$F(z, s, t) = C_i^k(s, t)z^M_i \exp\left(2\nu\left(w_{i,1}^k|s| + w_{i,2}^k|t|\right)\right).$$
From [3-4] in Proposition 2.3 we know that if $\frac{\partial F(z, s, t)}{\partial s} \neq 0$, then the following inequality holds
$$k_0\alpha_q < \alpha_1 M_{i,1}^k + \cdots + \alpha_q M_{i,q}^k,$$
and if $\frac{\partial F(z, s, t)}{\partial t} \neq 0$, then correspondingly we have
$$k_0\beta_p < \beta_1 M_{i,q+1}^k + \cdots + \beta_p M_{i,q+p}^k.$$  
(4.15)

Take $j \in \mathbb{N}^\nu$ and $l \in \mathbb{N}^p$. Write $\mathcal{J}(z) = F(z, s(z), t(z))$ where $s(z) = \Phi_+(z)$ and $t(z) = \Phi_-(z)$. Now from Proposition 2.3 we have
$$\frac{\partial |\mathcal{J}|}{\partial x^j} \leq O(|X|^k_1), \quad \frac{\partial |\mathcal{J}|}{\partial y^l} \leq O(|Y|^k_2),$$
(4.16)

where
$$k_1 > \alpha_1 M_{i,1}^k + \cdots + \alpha_q M_{i,q}^k - 2\nu w_{i,1}^k - |j|\alpha_q,$$
and
$$k_2 > \beta_1 M_{i,q+1}^k + \cdots + \beta_p M_{i,q+p}^k - 2\nu w_{i,2}^k - |l|\beta_p.$$  

Since $F(z, s, t)$ is a $2\nu$-tagged monomial function, we have that
$$w_{i,1}^k \leq K_1\left(M_{i,1}^k + \cdots + M_{i,p}^k\right), \quad w_{i,2}^k \leq K_2\left(M_{i,q+1}^k + \cdots + M_{i,q+p}^k\right).$$
Therefore, we obtain the following relations.
$$k_1 > (\alpha_1 - 2\nu K_1)M_{i,1}^k + \cdots + (\alpha_q - 2\nu K_1)M_{i,q}^k - |j|\alpha_q,$$
and
$$k_2 > (\beta_1 - 2\nu K_2)M_{i,q+1}^k + \cdots + (\beta_p - 2\nu K_2)M_{i,q+p}^k - |l|\beta_p.$$
With (4.14) and (4.15), it turns out that if \( \max\{|j|, |l|\} \leq k_0 \), then both \( k_1 \) and \( k_2 \) are positive. This implies that \( \mathcal{J} \) and therefore \( \Psi \) are \( C^{k_0} \).

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References


Appendix

This appendix consists of the proof to all the propositions given in the article.

A1. Proof of Proposition 2.2

Let \( v > 0 \), and \( \mu_1, \ldots, \mu_l \) be real numbers such that \( |\mu_j| < v j \), for \( j = 1, \ldots, l \). Let

\[
T_j : \mathbb{R} \rightarrow \mathbb{R}, \quad s \mapsto T_j(s) = \chi \ast \mu_1 \ast \cdots \ast \mu_l(s).
\]

To prove Proposition 2.2, it suffices to show the validity of the following statements.

[A-1] \( T_j(0) = \frac{d^{l-1}T_j}{dt^{l-1}}(0) = \cdots = \frac{d^{l-1}T_j}{dt^{l-1}}(0) = 0 \),

[A-2] \( \frac{d^{l}T_j}{dt^{l}}(0) \neq 0 \);

[A-3] The function

\[
C_j : \mathbb{R} \rightarrow \mathbb{R}, \quad s \mapsto e^{-2v|s|}T_j(s),
\]

together with their derivatives are bounded uniformly in \( l \). In other words, for each integer \( i \), there exists \( K_i > 0 \) such that

\[
\sup_{s \in \mathbb{R}} \left| \frac{d^i C_j}{ds^i}(s) \right| < K_i, \quad \text{i.e.}, \quad \left| \frac{d^i T_j}{ds^i}(s) \right| < K_i e^{2v|i|}.
\]
The properties of [A-1] and [A-2] can be easily verified by induction on the orders of the derivatives $T_i$ with respect to its arguments. To prove the validity of [A-3], we first notice the relation

$$C'_i(s) = \pm 2ihC_i + \mu C_i + e^{-2ih|s|}C_{i-1}.$$ 

Therefore, if the family $\{C_i\}_{i \in \mathbb{N}}$ is uniformly bounded, then the induction tells us that for each integer $i$, there exists $K_i > 0$ such that for all integer $l$, $|C_i| \leq K_i$. Thus, it is sufficient to show that there exists $K > 0$ such that for all integer $l$, $|C_l| \leq K$. Moreover, we shall only consider the case on $\mathbb{R}^+$. One can fill up the proof for $\mathbb{R}^-$ in the same way.

For each integer $l$ we put $K_l := \sup C_l$. Then $K_0 = 1$. In what follows we shall first show that the sequence $\{K_l\}_{l \in \mathbb{N}}$ is well defined (i.e. $K_l < \infty$ for all $l$) and then we prove that it decreases when $l$ is large. In fact, since $C_l \leq K_l$, by the definition of $T_i(s)$ we have

$$T_i(s) \leq K_i e^{2ivs}.$$ 

Moreover, from the following relations

$$T_{i+1}(s) = \int_0^s T_i(s-u)e^{iuis}du,$$

$$C_{i+1}(s) = \int_0^s T_i(s-u)e^{-2(l+1)us}e^{iuis}du,$$ 

we obtain that

$$C_{i+1}(s) \leq K_i \int_0^s e^{2ivs}e^{-2ivu}e^{-2(l+1)us}e^{iuis}du$$

$$\leq K_i e^{-2vs} \int_0^s e^{-2ivu}e^{iuis}du$$

$$\leq K_i e^{-2vs} \int_0^s e^{-2ivu}e^{i(l+1)us} \leq K_i e^{-2vs} \int_0^s e^{-v(l-1)us}du$$

This implies that

$$C_{i+1}(s) \leq K_i L_i,$$ 

(A4)

where $L_l := \sup_{s > 0} (e^{-2vs} \int_0^s e^{-v(l-1)us}du)$. Notice that for sufficiently large $l$, the quantity

$$L_{l+1} = \sup_{s > 0} \left( e^{-2vs} \int_0^t e^{-v(u-s)}du \right) = \sup_{s > 0} \left( e^{-2vs} \left[ \frac{1 - e^{-hv}}{hv} \right] \right)$$

is less than one. On the other hand, we have $K_0 = 1$,

$$C_1 \leq \sup_{s > 0} \left( e^{-vs} \int_0^t e^{-v(u-s)}du \right) = \frac{1}{4v}, \quad \text{i.e.} \quad K_1 = \frac{1}{4v}$$

$$C_2 \leq \sup_{s > 0} (K_1 se^{-2vs}) = \frac{K_1}{2e^v} < \infty.$$ 

Now with (5.3) it turns out that for $r$ large enough

$$C_{i+1} \leq K_{i+1} \leq L_l K_i \leq K_i.$$ 

(A5)

This means that the sequence $\{K_l\}_{l \in \mathbb{N}}$ is decreasing for large $l$, therefore the above facts altogether imply that for each $l$, $K_l < \infty$. Moreover, $\sup_{l \in \mathbb{N}} K_l < \infty$. This ends the proof of Proposition 2.2. $$\square$$
A2. Proof of Proposition 2.3
We shall adopt the following notation.
If \( a = (a_1, \ldots, a_N) \in \mathbb{R}^N \), then we denote by
\[
z^a = z_1^{a_1} \cdots z_N^{a_N}.
\] (A6)

If \( M = (M_{ij}) \) is an \( m \times N \) matrix, where \( M_{ij} \in \mathbb{N} \), then we denote by
\[
z^M = (z^{M_1}, \ldots, z^{M_N}),
\]
where \( M_j = (M_{j1}, \ldots, M_{jN}) \) and where \( z^M \) is defined through (A6). For example, if \( M = (M_{ij}) \) is an \( m \times 2 \) matrix, i.e. \( M = (M_1, \ldots, M_m)^T \) and \( u = (e^{[h]}, e^{[l]}) \), then
\[
\]

Given two vectors \( w = (w_1, \ldots, w_m) \) and \( v = (v_1, \ldots, v_m) \) in \( \mathbb{R}^m \), we formally write \( wv \) in the following way.
\[
wv = (w_1v_1, \ldots, w_mv_m) \in \mathbb{R}^m.
\]
Moreover, if \( J = (J_{ij}) \) is an \( m \times l \) matrix, then we formally write \( wJ \) as an \( m \times l \) matrix whose entries are given by
\[
wJ = (w_iJ_{ij}).
\]
The following conventional symbol still stands
\[
\nabla \cdot \frac{\partial}{\partial v} = \sum_{i=1}^{m} v_i \frac{\partial}{\partial v_i}.
\]
Finally, let \( f : \mathbb{R}^N \to \mathbb{R} \) be a smooth function and \( j = (j_1, \ldots, j_N) \in \mathbb{N}^N \). We put
\[
f^{[j]}(z) = \frac{\partial^{[j]} f}{\partial z^{[j]}}(z) = \frac{\partial^{[j]} f}{\partial z_1^{j_1} \cdots \partial z_N^{j_N}}(z).
\]
The Property [3-1] is obvious and hence its proof is omitted. To prove the validity of Property [3-2], we write
\[
f(z, s, t) = \sum_{[i]=1}^{\infty} e^{[i]}(s, t) z^{M^{[i]}} u^{W^{[i]}},
\] (A7)
\[
g(z, s, t) = \sum_{[i]=1}^{\infty} \tilde{e}^{[i]}(s, t) z^{\tilde{M}^{[i]}} \tilde{u}^{\tilde{W}^{[i]}},
\]
where \( e^{[i]}(s, t) \) and \( \tilde{e}^{[i]}(s, t) \) are some vector functions, \( M^{[i]} \) and \( \tilde{M}^{[i]} \) are \( m \times m \) matrices, \( W^{[i]} \) and \( \tilde{W}^{[i]} \) are \( m \times 2 \) matrices and \( u = (e^{[h]}, e^{[l]}) \). We then have
\[
f\tilde{g}(z, s, t) = \sum_{[i]+[j]=1}^{\infty} e^{[i]}(s, t) \tilde{e}^{[j]}(s, t) z^{M^{[i]+\tilde{M}^{[j]}}} u^{W^{[i]}+\tilde{W}^{[j]}}.
\]
Since for each integer \( l = 1, \ldots, m, \)
\[
|M^{[i]}| \geq |W^{[i]}| + 1, \quad |\tilde{M}^{[j]}| \geq |\tilde{W}^{[j]}| + 1,
\]
it follows that
\[
|M^{[i]}| + |\tilde{M}^{[j]}| \geq |W^{[i]}| + |\tilde{W}^{[j]}| + 2.
\]
Thus, property [3-2] follows.
Now we consider property [3-3]. Firstly, from (5.7) we have

$$H(z, s, t) = \sum_{j=1}^{\infty} c^{(j)}(s, t) \left( \sum_{j=1}^{\infty} c^{(j)}(s, t) z^{M^{(j)}} \right)^{M^{(j)}} u^{W^{(j)}}.$$  \hspace{1cm} (A8)

We point out the following property: If \( \{H_j\} \) is a sequence of \( \mathbb{R}^m \rightarrow \mathbb{R}^m \) functions \( H_j = (H_{1,j}, \ldots, H_{m,j}) \), \( N = (N_1, \ldots, N_m)^T \) is an \( n \times m \) matrix, then

$$\left( \sum_{j \in \mathbb{N}} H_j \right)^N = (h_1, \ldots, h_n)$$

where

$$h_l = \left( \sum_{j \in \mathbb{N}} H_j \right)^{N_l} = \left( \sum H_{1,j} \right)^{N_{l,1}} \cdots \left( \sum H_{m,j} \right)^{N_{l,m}} = \sum_{r=1}^{\infty} \sum_{(j_1, \ldots, j_r) \in N^r} A_s H_{1,j_1}^{N_{l,1}} \cdots H_{1,j_r}^{N_{l,m}} \cdots H_{m,j_r}^{N_{l,m}}$$

where \( A_s \) are constant, and for each \( l \in \{1, \ldots, n\} \),

$$N_{l,k,j_1} + \cdots + N_{l,k,j_r} = N_{l,k}, \quad k = 1, \ldots, m.$$  

Applying this property to the term \( (\sum_{j=1}^{\infty} c^{(j)}(s, t) z^{M^{(j)}} u^{W^{(j)}})^{M^{(j)}} \) which is denoted by \( (h_1, \ldots, h_N) \). Then for \( 1 \leq l \leq N \), \( h_l \) has the following form.

$$h_l = \sum A_{l,k}(s, t) z^{P_{l,k}} u^{Q_{l,k}}$$

where \( P_{l,k} \) and \( Q_{l,k} \) are some vectors derived from \( M^{(i)} \) and \( M^{(j)} \). It remains to show that for all \( 1 \leq l \leq p + q \)

$$|P_{l,k}| \geq |Q_{l,k}| + 1.$$  \hspace{1cm} (A9)

Based on the assumption that \( f \) and \( g \) are \( \nu \)-tagged function, this inequality in fact can be verified straightforwardly.

Now we prove [3-4]. We shall only prove [3-4], the proofs of [3-4-1] and [3-4-2] are similar to the proof of Lemma 1 and we leave this part to the reader.

To prove [3-4], take a function \( f \) that takes the form written in (2.11). Write

$$\tilde{f}(z, s, t) = (f(z, s, t), s, t).$$

We construct a sequence of function \( \{\tilde{g}^{(i)}\}_{i \in \mathbb{N}} \) where

$$\tilde{g}^{(i)}(z, s, t) = (g_1^{(i)}(z, s, t), \ldots, g_N^{(i)}(z, s, t))$$

such that the function \( \tilde{g}^{(i)}(z, s, t) = (g^{(i)}(z, s, t), s, t) \) satisfies the following. For each integer \( 1 \leq k \leq p + q \),

- \( g^{(i+1)}_k(z, s, t) - g^{(i)}_k(z, s, t) = h^{(i+1)}_k(z, s, t) \) with \( \lim_{s \to \infty} d_0(h^{(i+1)}_k) = \infty \).
- \( \tilde{g}^{(i)}(\tilde{f}(z, s, t)) = z + H^{(i)}(z, s, t) \) with \( \lim_{s \to \infty} d_0(H^{(i+1)}_k) = \infty \).
We set \( g_0(z, s, t) = z \). Write again
\[
f(z, s, t) = z + H_0(z, s, t), \quad \text{where } H_0 = \sum_{i \in \mathbb{N}} C^{(i)}(s, t)z^M u^W. \]

Write
\[
h^{(i)}(z, s, t) = -\sum_{i \in \mathbb{N}} C^{(i)}(s, t)z^M u^W. \]

Remark that for each integer \( i \) and for each \( k 
\[
(z + H_0(z, s, t))^M = z^M + G^{(i, 0)}_k(z, s, t)
\]
where \( d_0(G^{(i, 0)}_k) > d_1(z^M) \). We then get
\[
g_i \left( \tilde{g}(z, s, t) \right) = z - \sum_{i \in \mathbb{N}} C^{(i, 1)}(s, t)(z + H_0(z, s, t))^M u^W
\]
\[
= z + \sum_{i \in \mathbb{N}} C^{(i, 1)}(s, t)z^M u^W. \quad \text{(A10)}
\]

From [3-3] we already know that the expression written in (5.11) is a \( v \)-tagged function, the \( C^{(i, 1)} \)'s are smooth functions and for each integer \( k = 1, \ldots, p + q \),
\[
\min_{i \in \mathbb{N}} |M_k^i| < \min_{i \in \mathbb{N}} |M_k^{(i, 1)}|. 
\]
Assume now that \( h^{(1)}, \ldots, h^{(i)} \) has been constructed such that the \( g^{(i)} \)'s satisfy \( g^{(k + 1)} = g^{(k)} + h^{(k + 1)} \) and for all \( k = 1, \ldots, i - 1 \),
\[
g^{(k + 1)}(\tilde{g}(z, s, t)) = z + H^{(k + 1)}(z, s, t)
\]
where for each integer \( m = 1, \ldots, p + q \), \( d_0(H_m^{(k)}) < d_0(H_m^{(k + 1)}) \). Write now \( h^{(i + 1)} = -H^{(i)}(z, s, t) \). We therefore get
\[
g^{(i + 1)}(\tilde{g}(z, s, t)) = z + H^{(i)}(z, s, t) = H^{(i)} \left( z + H^{(i + 1)}(z, s, t), s, t \right) \]
\[
= z + H^{(i + 1)}(z, s, t) \quad \text{(A12)}
\]
where for each integer \( m = 1, \ldots, p + q \), \( d_0H^{(i + 1)}_m > d_0H^{(i)}_m \). Finally, assume that for each integer \( i \) and \( j \), \( z^M \) is a \( v \)-quasi-resonant monomial. Write
\[
g_j^{(i)}(z, s, t) = \sum_{i \in \mathbb{N}} C_j^{(i, k)}(s, t)z^M u^W. 
\]
By construction of the inverse function, we see for each integers \( i, i, k \) the monomial \( z^W \) is also a \( v \)-quasi-resonant term. This ends the proof of the proposition.

\textbf{A3. proof of Proposition 4.1}

We shall show that (4.10) holds with (4.12). The proof of (4.11) with (4.13) is completely similar.

\textbf{Proposition 5.1:} \textit{Let } \( H : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \text{ be a function having the form}
\[
H : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad (s, t) \mapsto e(s, t) \exp \left( rv(|t| + |s|) \right)
\]
and satisfying

\[ |H^{(0)}(s, t)| \leq \mathcal{O}(\exp(2\nu(|s| + |s|))) \]  

(A13)

for all \( j \in \mathbb{N}^2 \). Then the function \( h(z) = H(\Phi_+(z), \Phi_-(z)) \) satisfies the following inequalities:

\[ \frac{\partial^{|I|+|\Gamma|}}{\partial \bar{z}^I \partial z^\Gamma}(z) \leq \mathcal{O}(X^{-|I^+|\alpha_2 - 2\nu}) \mathcal{O}(Y^{-|\Gamma^-|\beta_2 - 2\nu}), \]

where \( I^+ \in \mathbb{N}^q \) and \( \Gamma^- \in \mathbb{N}^p \).

**Proof:** Let \( a \) and \( b \) be two integers and define the set \( \mathbb{N}_{(a)}^b \) as follows.

\[ \mathbb{N}_{(a)}^b = \{ i \in \mathbb{N} \mid \|i\| = i_1 + \cdots + i_b = a \}. \]

Now fix \( k \) and let \( I = (I^-, I^+) \in \mathbb{N}_{(k)}^q \), then

\[ h^{(0)}(z) = \sum_{i=1}^{k} \sum_{x \in \mathbb{N}_{(i)}} \sum_{L} \sum_{a, a^+} A_{x, a_1, \ldots, a_k} H(\Phi_+(z), \Phi_-(z)) \cdot \Phi_+^{(a_1^+)} \cdots \Phi_+^{(a_k^+)}, \]  

(A14)

where \( (a_j^-, a_j^+) = a_j \) (\( j = 1, \ldots, k \)), and

\[ L = \left\{ a_1, \ldots, a_k \mid a_1^-, \ldots, a_k^- \in \mathbb{N}_1^q \}, a_1^+, \ldots, a_k^+ \in \mathbb{N}_1^p \right\}. \]

\( A_{x, a_1, \ldots, a_k} \) are constants. This property can be shown by induction on \( k \).

On the other hand, due to (5.13), we have that

\[ |H^{(k)}(\Phi_+(z), \Phi_-(z))| \leq \mathcal{O}(X^{-2\nu}) \mathcal{O}(Y^{-2\nu}). \]

By the following lemma, we know that for all \( a_j^+ \in \mathbb{N}_{(k)}^q \) and \( a_j^- \in \mathbb{N}_{(k)}^p \),

\[ |\Phi_+^{(a_j^+)}(z)| \leq \mathcal{O}(X^{-|a_j^+|\alpha_2}) \quad \text{and} \quad |\Phi_-^{(a_j^-)}(z)| \leq \mathcal{O}(Y^{-|a_j^-|\beta_2}). \]

Therefore,

\[ |\Phi_+^{(a_1^+)} \cdots \Phi_+^{(a_k^+)}| \leq \mathcal{O}(X^{-|I^+|\alpha_2}) \mathcal{O}(Y^{-|\Gamma^-|\beta_2}), \]

and we obtain the desired estimation.

**Lemma 5.1:** The map \( (\Phi = \Phi_+, \Phi_-) \) satisfies the following inequalities:

\[ \left| \frac{\partial^{|I|}}{\partial \bar{z}^I \partial z^\Gamma}(\Phi_{\pm}) \right| \leq \mathcal{O}(X^{-|I^+|\alpha_2}), \quad |i| = k, \]

\[ \left| \frac{\partial^{|I|}}{\partial \bar{z}^I \partial z^\Gamma}(\Phi_{\pm}) \right| \leq \mathcal{O}(Y^{-|\Gamma^-|\beta_2}), \quad |i| = k \]

for all \( j \in \mathbb{N}^q \) and \( i \in \mathbb{N}^p \).

This lemma can be proven by considering quasi-homogenous polar coordinates. A complete proof of this lemma is given in [8], See also [9].

We are now in position to prove (4.10) and (4.12). For each integer \( i \), consider the functions

\[ P(z) = x^i, \]

\[ J(z) = \Phi_+(\Phi_+(z)) \exp(2w_1|\Phi_+(z)|). \]
Then by the Leibnitz rule, we have
\[
\frac{\partial^{[i]}(PJ)}{\partial x^i}(x) = \sum_{j=0}^{[i]} \sum_{I^{(i)}, L^{(i)} \in \mathbb{N}^i_{[i]} \cap \mathbb{N}^j_{[i]-j}} A_{J^{(i)}, L^{(i)}} P^{(i)}(z) J^{(i)}(z)
\]  
(A15)

where $A_{J^{(i)}, L^{(i)}}$'s are some integers depending only on $i$.

Put

$I^{[1]} = (l_1^{[1]}, \ldots, l_q^{[1]}), \ I^{[2]} = (l_1^{[2]}, \ldots, l_q^{[2]}).$

It is easy to see that

\[
\left| P^{(i)}(z) \right| \leq O(X^{k_1}),
\]  
(A16)

where

\[
k_1 = (\alpha, n^+ - I^{[1]}) = \alpha_1 (n_1^+ - l_1^{[1]}) + \cdots + \alpha_q (n_q^+ - l_q^{[1]}).
\]

With Proposition 5.1 we also have

\[
\left| J^{(i)}(z) \right| \leq O(X^{-k_2})
\]  
(A17)

where

\[
k_2 = \alpha_q |I^{[2]}| + 2w_1.
\]

Since for all integer $1 \leq i \leq q$, $\alpha_i \leq \alpha_q$, we have that

\[
K_1 = k_1 - k_2 > -\alpha_q |I| - 2uw_1 + (\alpha, n^+)
\]

We end the proof of Proposition 4.1. \qed