Heron triangles
which cannot be decomposed into two integer right triangles

Paul Yiu
Department of Mathematical Sciences,
Florida Atlantic University,
Boca Raton, Florida 33431

yiu@fau.edu

41st Meeting of Florida Section of
Mathematical Association of America
Flordia Southern College,
Lakeland, Florida
February 15-16, 2008

Abstract

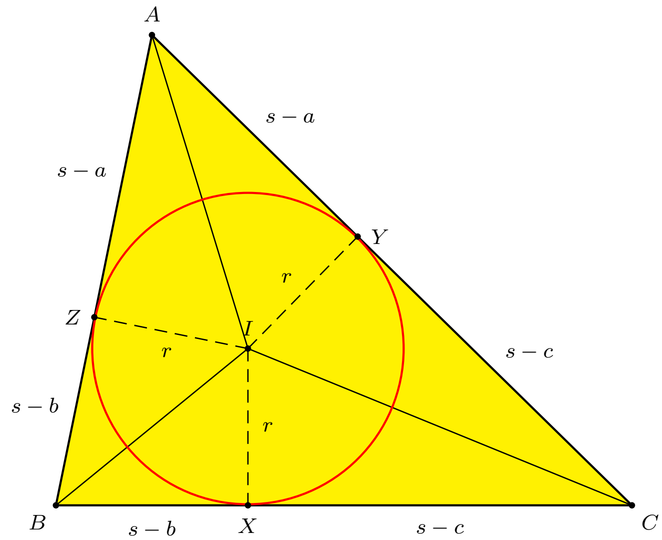
A Heron triangle is one whose sides and area are integers. While it is quite easy to construct Heron triangles by joining two integer right triangles along a common side, there are some which cannot be so obtained. For example, the Heron triangle $(25, 34, 39)$ has integer area 420 but no integer altitude. In this talk, a systematic construction of such indecomposable Heron triangles will be presented.

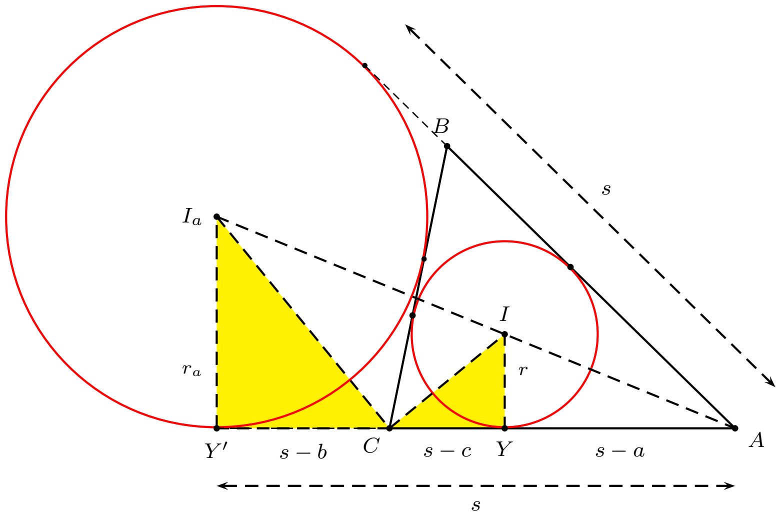
CONTENTS

1. Heron's formula for the area of a triangle

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s := \frac{1}{2}(a+b+c)$.



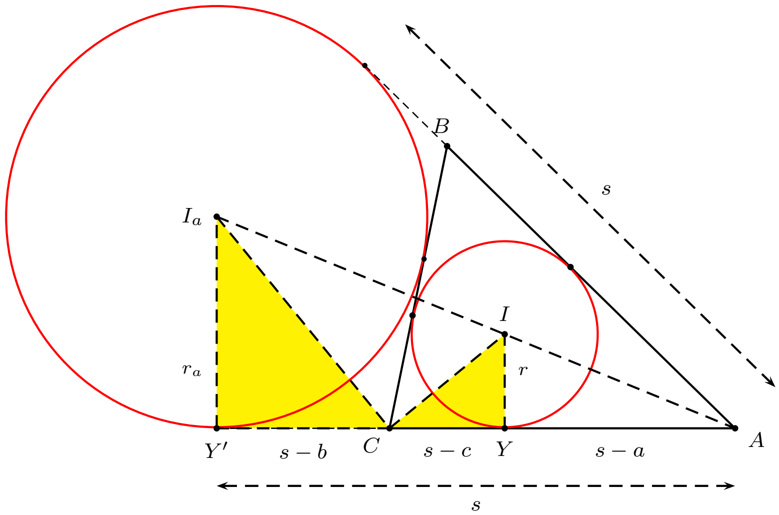


- $\triangle = rs$.
- From the similarity of triangles AIY and AI_aY',

$$\frac{r}{r_a} = \frac{s-a}{s}.$$

- From the similarity of triangles CIY and I_aCY',

$$r \cdot r_a = (s-b)(s-c).$$



From

$$\frac{r}{r_a} = \frac{s-a}{s}, \qquad r \cdot r_a = (s-b)(s-c),$$

we obtain

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$
$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}.$$

Examples

(1)

a	b	c	s	$s-a$	$s-b$	$s-c$	$\triangle$
3	4	5	$6 = 2\cdot 3$	3	2	1	$2\cdot 3 = 6$

(2)

a	b	c	s	$s-a$	$s-b$	$s-c$	$\triangle$
13	14	15	$21 = 3\cdot 7$	$8 = 2^3$	7	$6 = 2\cdot 3$	$2^2\cdot 3\cdot 7 = 84$

(3)

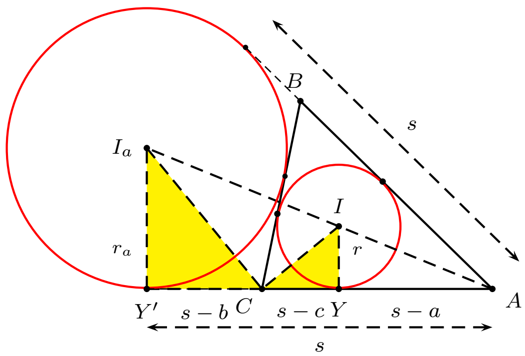
a	b	c	s	$s-a$	$s-b$	$s-c$	$\triangle$
25	34	39	$49 = 7^2$	$24 = 2^3\cdot 3$	$15 = 3\cdot 5$	$10 = 2\cdot 5$	$2^2\cdot 3\cdot 5\cdot 7 = 420$

How can one construct Heron triangles?

(1) Put $u = s - a$, $v = s - b$, and $w = s - c$. Then $s = u + v + w$.

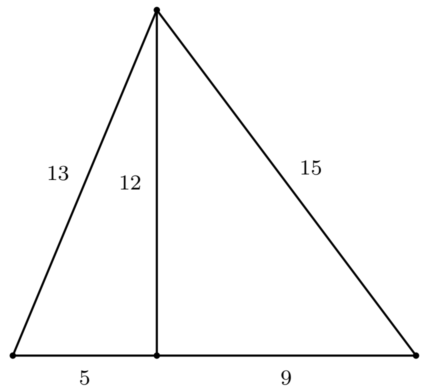
We require

$$uvw(u + v + w) = \square.$$



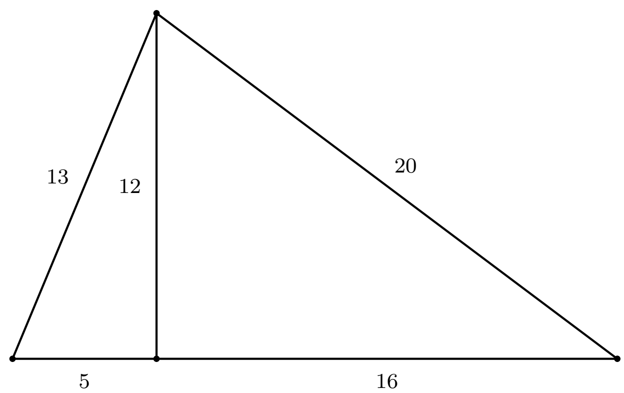
(2) From three positive rational numbers t_1, t_2, t_3 satisfying $t_1t_2 + t_2t_3 + t_3t_1 = 1$. (Section 2 below).

(3) A more naïve approach is to put two integer right triangles together along a common side:



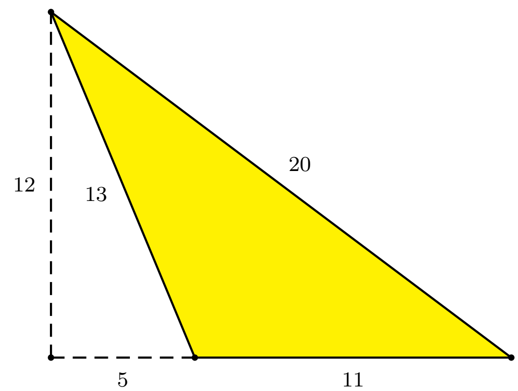
$$(13, 14, 15; 84) = (12, 5, 13; 30) \cup (12, 9, 15; 54).$$

By joining $4(3,4,5)$ and $(5,12,13)$ along the common side 12, we also get $(13,20,21;126)$:



$$(13,20,21;126) = (12,5,13;30) \cup (12,16,20;96).$$

We may also cut out a small Pythagorean triangle from a larger one. For example,



$$(11, 13, 20; 66) = (12, 16, 20; 96) \setminus (12, 5, 13; 30).$$

Does every Heron triangle arise in this way?

We say that a Heron triangle is **decomposable** if it can be obtained by joining two Pythagorean triangles along a common side, or by excising a Pythagorean triangle from a larger one.

Clearly, a Heron triangle is decomposable if and only if it has an **integer height** (which is not a side of the triangle).

The Heron triangle $(25, 34, 39; 420)$ is not decomposable because it does not have an integer height. Its heights are

$$\frac{840}{25} = \frac{168}{5}, \; \frac{840}{34} = \frac{420}{17}, \; \frac{840}{39} = \frac{280}{13},$$

none an integer.

This example was first obtained by Fitch Cheney.

W. F. Cheney, Heronian triangles, AMER. MATH. MONTHLY, 36 (1929) 22–28.

Cheney was perhaps more famous for his card trick described in

Michael Kleber, The best card trick, THE MATHEMATICAL INTELLIGENCER, 24 (2002) Number 1, 9–11.

Kleber wrote

> [William Fitch Cheney, Jr.] was born in San Francisco in 1894, ... After receiving his B.A. and M.A. from the University of California in 1916 and 1917, ...[i]n 1927 he earned the first math Ph.D. ever awarded by MIT. ...Fitch [taught] at the University of Hartford ...until his death in 1974.

Cheney's second example of indecomposable Heron triangle:

$$(39, 58, 95; 456).$$

a	b	c	s	$s-a$	$s-b$	$s-c$	$\triangle$
39	58	95	$96 = 2^5 \cdot 3$	$57 = 3 \cdot 19$	$38 = 2 \cdot 19$	1	$2^3 \cdot 3 \cdot 19 = 456$

Heights:

$$\frac{304}{13}, \frac{456}{29}, \frac{48}{5}.$$

Nowadays, it is much easier to do a computer search.

Smallest indecomposable Heron triangle:

$$(5, 29, 30; 72).$$

a	b	c	s	$s-a$	$s-b$	$s-c$	$\triangle$
5	29	30	$32 = 2^5$	$27 = 3^3$	3	2	$2^3 \cdot 3^2 = 72$

Smallest indecomposable **acute** Heron triangle:

$$(15, 34, 35; 252).$$

a	b	c	s	$s-a$	$s-b$	$s-c$	$\triangle$
15	34	35	$42 = 2 \cdot 3 \cdot 7$	$27 = 3^3$	$8 = 2^3$	7	$2^2 \cdot 3^2 \cdot 7 = 252$

How can one construct examples of indecomposable Heron triangles?

Restrict to **primitive** ones in which the sides do not have common divisors.

We give a systematic construction of primitive Heron triangles and examine the condition for the indecomposability.

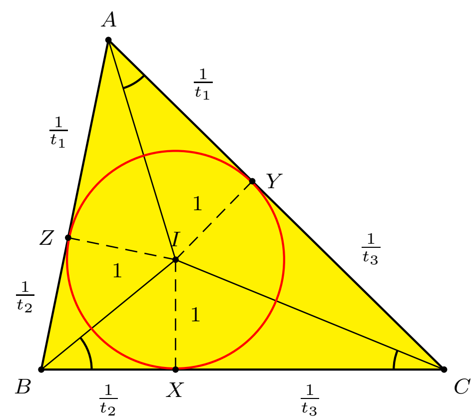
2. Construction of primitive Heron triangles

The **similarity class** of a Heron triangle is determined by three positive **rational** numbers

$$t_1 = \tan\frac{A}{2}, \qquad t_2 = \tan\frac{B}{2}, \qquad t_3 = \tan\frac{C}{2}.$$

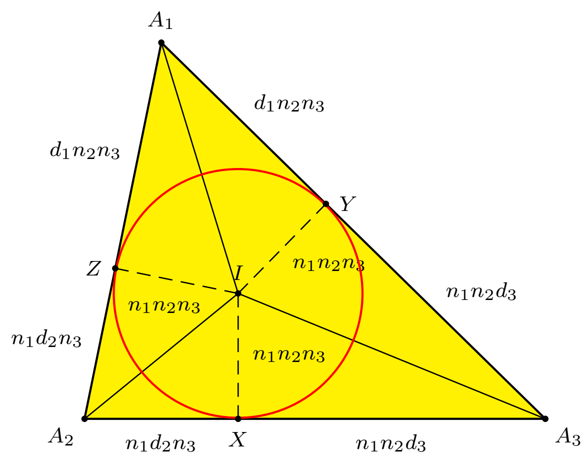
Since $\frac{A}{2} + \frac{B}{2} + \frac{C}{2} = \frac{\pi}{4}$, these numbers satisfy

$$t_1t_2 + t_2t_3 + t_3t_1 = 1.$$



By clearing denominators, we obtain Heron triangles.

Putting $t_i = \frac{n_i}{d_i}$, $i = 1, 2, 3$, with $\gcd(n_i, d_i) = 1$, and magnifying by $n_1n_2n_3$ times, we have the Heron triangle



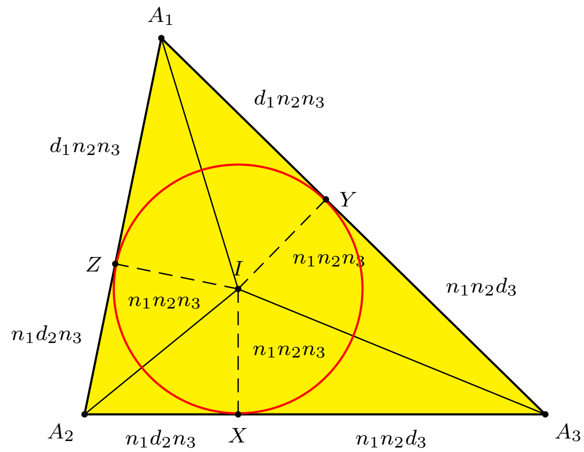
Here,

$$n_1n_2d_3 + n_1d_2n_3 + d_1n_2n_3 = d_1d_2d_3. \quad (1)$$

This is a Heron triangle with sides

$a = n_1(d_2n_3+n_2d_3),\ b = n_2(d_3n_1+n_3d_1),\ c = n_3(d_1n_2+n_1d_2),$

semiperimeter $s = n_1n_2d_3 + n_1d_2n_3 + d_1n_2n_3 = d_1d_2d_3$
and area $\triangle = n_1d_1n_2d_2n_3d_3$.



A **primitive** Heron triangle Γ_0 results by dividing by the sides by $g := \gcd(a, b, c)$.

3. Decomposability of primitive Heron triangles

Theorem 1. *A primitive Heron triangle can be decomposed into two Pythagorean components in at most one way,* i.e., *it can have at most one integer height.*

Proof. This follows from three propositions.

(1) A primitive Pythagorean triangle is indecomposable. [1]

(2) A primitive, isosceles Heron triangle is decomposable, the only decomposition being into two congruent Pythagorean triangles. [2]

(3) If a non-Pythagorean Heron triangle has two integer heights, then it cannot be primitive. [3]

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[1]Proof of (1). We prove this by contradiction. A Pythagorean triangle, if decomposable, is partitioned by the altitude on the hypotenuse into two similar but *smaller* Pythagorean triangles. None of these, however, can have all sides of integer length by the primitivity assumption on the original triangle.

[2]Proof of (2). The triangle being isosceles and Heron, the perimeter and hence the base must be even. Each half of the isosceles triangle is a (primitive) Pythagorean triangle, $(m^2-n^2, 2mn, m^2+n^2)$, with m, n relatively prime, and of different parity. The height on each slant side of the isosceles triangle is

$$\frac{2mn(m^2-n^2)}{m^2+n^2},$$

which clearly cannot be an integer. This shows that the only way of decomposing a primitive isosceles triangle is into two congruent Pythagorean triangles.

[3]Proof of (3). Let $(a, b, c; \triangle)$ be a Heron triangle, not containing any right angle. Suppose the heights on the sides b and c are integers. Clearly, b and c cannot be relatively prime, for otherwise,

□

the heights of the triangle on these sides are respectively ch and bh, for some integer h. This is impossible since, the triangle not containing any right angle, the height on b must be less than c, Suppose therefore $\gcd(b,c) = g > 1$. We write $b = b'g$ and $c = c'g$ for relatively prime integers b' and c'. If the height on c is h, then that on the side b is $\frac{ch}{b} = \frac{c'h}{b'}$. If this is also an integer, then h must be divisible by b'. Replacing h by $b'h$, we may now assume that the heights on b and c are respectively $c'h$ and $b'h$. The side c is divided into $b'k$ and $\pm(c - b'k) \neq 0$, where $g^2 = h^2 + k^2$. It follows that

$$\begin{aligned} a^2 &= (b'h)^2 + (c'g - b'k)^2 \\ &= b'^2(h^2 + k^2) + c'^2g^2 - 2b'c'gk \\ &= g[g(b'^2 + c'^2) - 2b'c'k] \end{aligned}$$

From this it follows that g divides a^2, and every prime divisor of g is a common divisor of a, b, c. The Heron triangle cannot be primitive.

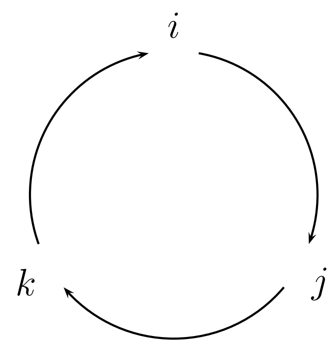
4. Triple of simplifying factors (for the similarity class of a Heron triangle)

Unless explicitly stated otherwise, whenever the three indices i, j, k appear altogether in an expression or an equation, they are taken as a *permutation* of the indices 1, 2, 3.

Note that from

$$t_1 t_2 + t_2 t_3 + t_3 t_1 = 1,$$

any one of t_i, t_j, t_k can be expressed in terms of the remaining two.



In the process of expressing $t_i = \frac{n_i}{d_i}$ in terms of $t_j = \frac{n_j}{d_j}$ and $t_k = \frac{n_k}{d_k}$ from

$$t_1t_2 + t_2t_3 + t_3t_1 = 1,$$

we encounter certain **cancellations**. Namely,

$$t_i = \frac{1 - t_jt_k}{t_j + t_k} = \frac{d_jd_k - n_jn_k}{n_jd_k + d_jn_k}$$

is simplified by canceling the **gcd**

$$g_i := \gcd(d_jd_k - n_jn_k, n_jd_k + d_jn_k)$$

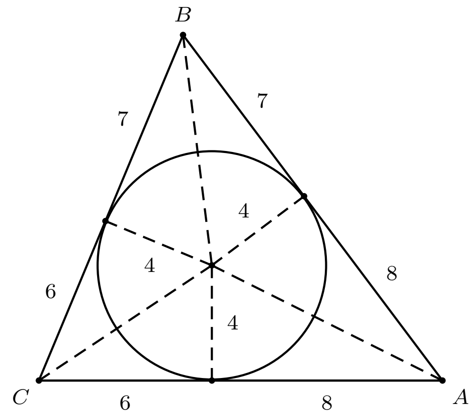
from the numerator and denominator. We call this g_i the **simplifying factor** of t_i from t_j and t_k:

$$\begin{aligned} g_in_i &= d_jd_k - n_jn_k, \\ g_id_i &= d_jn_k + n_jd_k. \end{aligned}$$

Likewise, there are simplifying factors g_j and g_k.

(g_1, g_2, g_3) is called the **triple of simplifying factors** for the numbers (t_1, t_2, t_3), or of the similarity class of triangles they define.

Example 1. For the $(13, 14, 15; 84)$, we have $t_1 = \frac{1}{2}$, $t_2 = \frac{4}{7}$ and $t_3 = \frac{2}{3}$.



$$t_1 = \frac{1 - t_2 t_3}{t_2 + t_3} = \frac{7 \cdot 3 - 4 \cdot 2}{7 \cdot 2 + 4 \cdot 3} = \frac{13}{26} = \frac{1}{2} \implies g_1 = 13,$$
$$t_2 = \frac{1 - t_3 t_1}{t_3 + t_1} = \frac{3 \cdot 2 - 2 \cdot 1}{3 \cdot 1 + 2 \cdot 2} = \frac{4}{7} = 1 \implies g_2 = 1,$$
$$t_3 = \frac{1 - t_1 t_2}{t_1 + t_2} = \frac{2 \cdot 7 - 1 \cdot 4}{2 \cdot 4 + 7 \cdot 1} = \frac{10}{15} = \frac{2}{3} \implies g_3 = 5.$$

Example 2. For the indecomposable Heron triangle $(25, 34, 39; 420)$ (Cheney's example),

a	b	c	s	$s-a$	$s-b$	$s-c$	$\triangle$
25	34	39	$49 = 7^2$	$24 = 2^3 \cdot 3$	$15 = 3 \cdot 5$	$10 = 2 \cdot 5$	$2^2 \cdot 3 \cdot 5 \cdot 7 = 420$

we have $r = \frac{\triangle}{s} = \frac{60}{7}$,

$$t_1 = \frac{r}{s-a} = \frac{5}{14}, \quad t_2 = \frac{r}{s-b} = \frac{4}{7}, \quad t_3 = \frac{r}{s-c} = \frac{6}{7}.$$

$$t_1 = \frac{1 - t_2 t_3}{t_2 + t_3} = \frac{7 \cdot 7 - 4 \cdot 6}{4 \cdot 7 + 6 \cdot 7} = \frac{25}{70} = \frac{5}{14} \implies g_1 = 5,$$

$$t_2 = \frac{1 - t_3 t_1}{t_3 + t_1} = \frac{7 \cdot 14 - 6 \cdot 5}{6 \cdot 14 + 5 \cdot 7} = \frac{68}{119} = \frac{4}{7} \implies g_2 = 17,$$

$$t_3 = \frac{1 - t_1 t_2}{t_1 + t_2} = \frac{14 \cdot 7 - 5 \cdot 4}{5 \cdot 7 + 14 \cdot 4} = \frac{78}{91} = \frac{6}{7} \implies g_3 = 13.$$

The simplifying factors are $(g_1, g_2, g_3) = (5, 17, 13)$.

5. Gaussian integers

We shall associate with each positive rational number

$$t = \frac{n}{d}, \qquad \gcd(n, d) = 1,$$

a **primitive, positive Gaussian integer**

$$z(t) := d + n\sqrt{-1} \in \mathbb{Z}[\sqrt{-1}].$$

Here, we say that a Gaussian integer $x + y\sqrt{-1}$ is

- **primitive** if x and y are relatively prime, and
- **positive** if both x and y are positive.

The **norm** of the Gaussian integer $z = x + y\sqrt{-1}$ is the integer

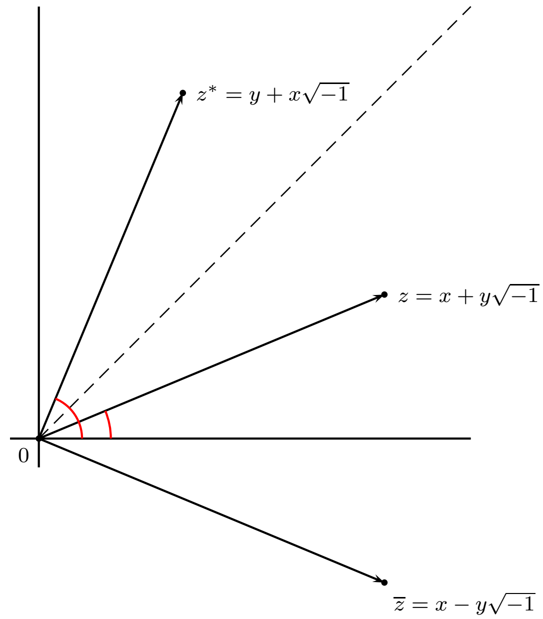
$$N(z) := x^2 + y^2.$$

The **argument** of a Gaussian integer $z = x + y\sqrt{-1}$ is the unique real number $\phi = \phi(z) \in [0, 2\pi)$ defined by

$$\cos\phi = \frac{x}{\sqrt{x^2 + y^2}}, \qquad \sin\phi = \frac{y}{\sqrt{x^2 + y^2}}.$$

We also consider the **complement** of $z = x + \sqrt{-1}y$:

$$z^* := y + x\sqrt{-1} = \sqrt{-1} \cdot \overline{z}.$$



$$\phi(z) + \phi(z^*) = \frac{\pi}{2}.$$

Some basic facts about the Gaussian integers

(1) The units of $\mathbb{Z}[\sqrt{-1}]$ are precisely ± 1 and $\pm\sqrt{-1}$.

(2) Multiplicativity of norm: $N(z_1 z_2) = N(z_1)N(z_2)$.

(3) An odd (rational) prime number p ramifies into two nonassociate primes $\pi(p)$ and $\overline{\pi(p)}$ in $\mathbb{Z}[\sqrt{-1}]$, namely,

$$p = \pi(p)\overline{\pi(p)} \quad \text{if and only if} \quad p \equiv 1 \bmod 4.$$

This is a consequence of Fermat's 2-square theorem: A odd prime number p is a sum of two squares in a unique way if and only if $p \equiv 1 \bmod 4$.

(4) Let $g > 1$ be an odd number. There is a **primitive** Gaussian integer θ satisfying $N(\theta) = g$ if and only if each prime divisor of g is congruent to 1 (mod 4).

6. Heron triangles and Gaussian integers

Corresponding to the rational numbers $t_i = \frac{n_i}{d_i}$, we consider the Gaussian integer $z_i = d_i + \sqrt{-1}n_i$.

The relations

$$g_i n_i = d_j d_k - n_j n_k, \qquad g_i d_i = d_j n_k + n_j d_k,$$

can be rewritten as

$$g_i z_i = \sqrt{-1} \cdot \overline{z_j z_k} = (z_j z_k)^*.$$

Lemma 2. $N(z_i) = g_j g_k$.

Proof. From $g_i z_i = (z_j z_k)^*$, we have

$$g_i^2 N(z_i) = N((z_j z_k)^*) = N(z_j z_k) = N(z_j)N(z_k).$$

Similarly,

$$g_j^2 N(z_j) = N(z_k)N(z_i) \quad \text{and} \quad g_k^2 N(z_k) = N(z_i)N(z_j).$$

Multiplying these latter two, and simplifying, we obtain

$$g_j^2 g_k^2 = N(z_i)^2 \implies N(z_i) = g_j g_k.$$

□

Lemma 2. $N(z_i) = g_j g_k$.

Proposition 3.

(1) g_i *is a common divisor of* $N(z_j)$ *and* $N(z_k)$.

(2) *At least two of* g_i, g_j, g_k *exceed 1.*

(3) g_i *is even if and only if all* n_j, d_j, n_k *and* d_k *are odd.*

(4) *At most one of* g_i, g_j, g_k *is even, and none of them is divisible by 4.*

(5) g_i *is prime to each of* n_j, d_j, n_k, *and* d_k.

(6) *Each odd prime divisor of* g_i, $i = 1, 2, 3$, *is congruent to* $1 \pmod 4$.

Proposition 4. $\gcd(g_1, g_2, g_3) = 1$.

Proof. (1) follows easily from Lemma 2.

(2) Suppose $g_1 = g_2 = 1$. Then, $N(z_3) = 1$, which is clearly impossible.

(3) is clear from the relation (**??**).

(4) Suppose g_i is even. Then n_j, d_j, n_k, d_k are all odd. This means that g_i, being a divisor of $N(z_j) = d_j^2 + n_j^2 \equiv 2 \pmod 4$, is not divisible by 4. Also, $d_j d_k - n_j n_k$ and $n_j d_k + d_j n_k$ are both even, and

$$\begin{aligned} & (d_j d_k - n_j n_k) + (n_j d_k + d_j n_k) \\ = \; & (d_j + n_j)(d_k + n_k) - 2n_j n_k \\ \equiv \; & 2 \pmod 4, \end{aligned}$$

it follows that one of them is divisible by 4, and the other is 2 (mod 4). After cancelling the common divisor 2, we see that exactly one of n_i and d_i is odd. This means, by (c), that g_j and g_k cannot be odd.

(5) If g_i and n_j admit a common prime divisor p, then p divides both n_j and $n_j^2 + d_j^2$, and hence d_j as well, contradicting the assumption that $d_j + n_j\sqrt{-1}$ be primitive.

(6) is a consequence of Proposition **??**. □

Proof of Proposition 4. We shall derive a contradiction by assuming a common rational prime divisor $p \equiv 1$ (mod 4) of g_i, g_j, g_k, with *positive* exponents r_i, r_j, r_k in their prime factorizations. By the relation (**??**), the product $z_j z_k$ is divisible by the *rational* prime power p^{r_i}. This means that the primitive Gaussian integers z_j and z_k should contain in their prime factorizations powers of the distinct primes $\pi(p)$ and $\overline{\pi(p)}$. The same reasoning also applies to each of the pairs $(z_k,\ z_i)$ and $(z_i,\ z_j)$, so that z_k and z_i (respectively z_i and z_j) each contains one of the non - associate Gaussian primes $\pi(p)$ and $\overline{\pi(p)}$ in their factorizations. But then this means that z_j and z_k are divisible by the *same* Gaussian prime, a contradiction.

7. Construction of Heron triangles with given simplifying factors

Example 3. $g_1 = 17$, $g_2 = 13$, $g_3 = 5$.

$N(z_1) = 13\cdot 5 = 65$, $N(z_2) = 5\cdot 17 = 85$, $N(z_3) = 17\cdot 13 = 221$.

$$\begin{array}{lllll} z_1: & 1+8\sqrt{-1} & 4+7\sqrt{-1} & 7+4\sqrt{-1} & 8+\sqrt{-1} \\ z_2: & 2+9\sqrt{-1} & 6+7\sqrt{-1} & 7+6\sqrt{-1} & 9+\sqrt{-1} \end{array}$$

$z_3 = \frac{1}{g_3}(z_1z_2)^*$	$2+9\sqrt{-1}$	$6+7\sqrt{-1}$	$7+6\sqrt{-1}$	$9+\sqrt{-1}$
$1+8\sqrt{-1}$	$5-14\sqrt{-1}$	$11-10\sqrt{-1}$		
$4+7\sqrt{-1}$	$10-11\sqrt{-1}$	$14-5\sqrt{-1}$		
$7+4\sqrt{-1}$			$14+5\sqrt{-1}$	$10+11\sqrt{-1}$
$8+\sqrt{-1}$			$11+10\sqrt{-1}$	$5+14\sqrt{-1}$

There are four classes of rational triangles with triple of simplifying factors $(17, 13, 5)$:

$$(t_1, t_2, t_3) = \left(\frac{4}{7}, \frac{6}{7}, \frac{5}{14}\right), \left(\frac{4}{7}, \frac{2}{9}, \frac{11}{10}\right), \left(\frac{1}{8}, \frac{6}{7}, \frac{10}{11}\right), \left(\frac{1}{8}, \frac{2}{9}, \frac{14}{5}\right).$$

Primitive Heron triangles with triple of simplifying factors $(17, 13, 5)$:

t_1	t_2	t_3	$(a, b, c; \triangle)$
$\frac{4}{7}$	$\frac{6}{7}$	$\frac{5}{14}$	$(34, 39, 25; 420)$
$\frac{1}{8}$	$\frac{2}{9}$	$\frac{5}{14}$	$(68, 117, 175; 2520)$
$\frac{1}{8}$	$\frac{6}{7}$	$\frac{10}{11}$	$(68, 273, 275; 9240)$
$\frac{4}{7}$	$\frac{2}{9}$	$\frac{11}{10}$	$(238, 117, 275; 13860)$

Theorem 5. *Let g_1, g_2, g_3 be* odd *numbers satisfying the following conditions.*

(i) *At least two of g_1, g_2, g_3 exceed 1.*

(ii) *The prime divisors of g_i, $i = 1, 2, 3$, are all congruent to 1 (mod 4).*

(iii) $\gcd(g_1, g_2, g_3) = 1.$

Suppose g_1, g_2, g_3 together contain λ distinct rational (odd) prime divisors. Then there are $2^{\lambda-1}$ distinct, primitive Heron triangles with simplifying factors (g_1, g_2, g_3).

Proof. Suppose (g_1, g_2, g_3) satisfies these conditions. By (ii), there are primitive Gaussian integers θ_i, $i = 1, 2, 3$, such that $g_i = N(\theta_i)$. Since $\gcd(g_1, g_2, g_3) = 1$, if a rational prime $p \equiv 1$ (mod 4) divides g_i and g_j, then, in the ring $\mathbb{Z}[\sqrt{-1}]$, the prime factorizations of θ_i and θ_j contain powers of the same Gaussian prime π or $\overline{\pi}$.

Therefore, if g_1, g_2, g_3 together contain λ rational prime divisors, then there are 2^λ choices of the triple of primitive Gaussian integers $(\theta_1,\ \theta_2,\ \theta_3)$, corresponding to a choice between the Gaussian primes $\pi(p)$ and $\overline{\pi(p)}$ for each of these rational primes. Choose units ϵ_1 and ϵ_2 such that $z_1 = \epsilon_1\theta_2\overline{\theta_3}$ and $z_2 = \epsilon_2\theta_3\overline{\theta_1}$ are positive.

Two positive Gaussian integers z_1 and z_2 define a positive Gaussian integer z_3 via $g_3 z_3 = (z_1 z_2)^*$ if and only if

$$0 < \phi(z_1) + \phi(z_2) < \frac{\pi}{2}. \tag{2}$$

Since $\phi(z_1^*) + \phi(z_2^*) = \pi - (\phi(z_1) + \phi(z_2))$, it follows that exactly one of the two pairs (z_1, z_2) and (z_1^*, z_2^*) satisfies condition (2). There are, therefore, $2^{\lambda-1}$ Heron triangles with (g_1, g_2, g_3) as simplifying factors. □

8. Decomposability of Heron triangles in terms of triple of simplifying factors

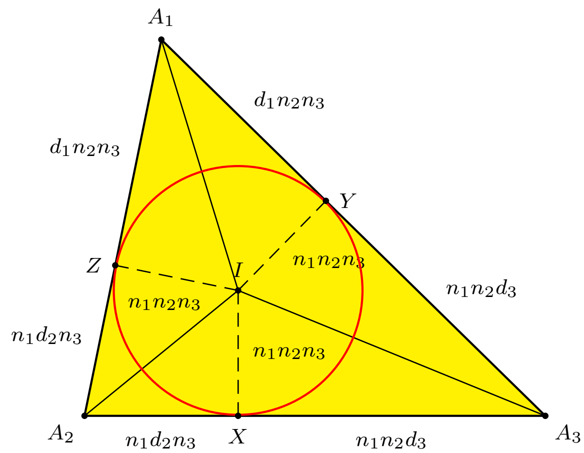
Proposition 6. *A Heron triangle is Pythagorean if and only if its triple of simplifying factors is of the form* $(1, 2, g)$*, for an odd number* g *whose prime divisors are all of the form* $4m + 1$.

Proof. If the Heron triangle contains a right angle, we may take $t_3 = \tan\frac{\pi}{4} = 1$ so that $g_1 g_2 = N(1 + \sqrt{-1}) = 2$. From this the numbers g_1 and g_2 must be 1 and 2 in some order.

Conversely, if $g_1 = 1$ and $g_2 = 2$, then $N(z_3) = 2 \implies z_3 = 1 + \sqrt{-1}$, $t_3 = 1$, and $C = 2\arctan 1 = \frac{\pi}{2}$. □

Here is the main theorem on indecomposable Heron triangles.

Consider the Heron triangle Γ again.



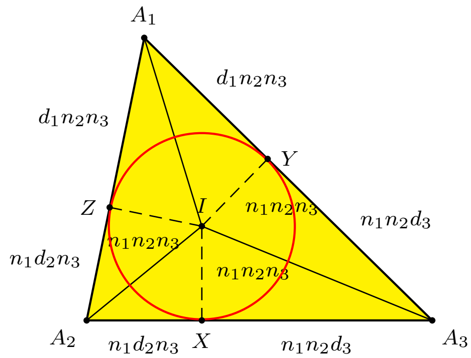
$$a = n_1(d_2n_3 + n_2d_3) = g_1n_1d_1,$$
$$b = n_2(d_3n_1 + n_3d_1) = g_2n_2d_2,$$
$$c = n_3(d_1n_2 + n_1d_2) = g_3n_3d_3.$$

Proposition 7. $\gcd(a, b, c) = \gcd(n_1d_1, n_2d_2, n_3d_3)$.

Proof. For $i = 1, 2, 3$, $a_i = g_in_id_i$. $\square$

Proposition 8. *The Heron triangle* Γ *(assumed non-Pythagorean) is indecomposable if and only if each of the simplifying factors* g_i, $i = 1, 2, 3$, *contains an odd prime divisor.*

Proof. We first consider the triangle Γ:



Since Γ has area $\triangle = n_1d_1n_2d_2n_3d_3$, the height on the side $a_i = g_in_id_i$ is given by

$$h_i = \frac{2n_jd_jn_kd_k}{g_i}.$$

Since the triangle does not contain a right angle, it is indecomposable if and only if none of the heights h_i, $i = 1, 2, 3$, is an integer. By Proposition 3(4), this is the case if and only if each of g_1, g_2, g_3 contains an odd prime divisor. $\square$

Theorem 9. *The* **primitive** *Heron triangle* Γ_0 *with half-tangents* (t_1, t_2, t_3) *(assumed non-Pythagorean) is indecomposable if and only if each of the simplifying factors* g_i*,* $i = 1, 2, 3$*, contains an odd prime divisor.*

Proof. The sides (and hence also the heights) of Γ_0 are $\frac{1}{g}$ times those of Γ, where

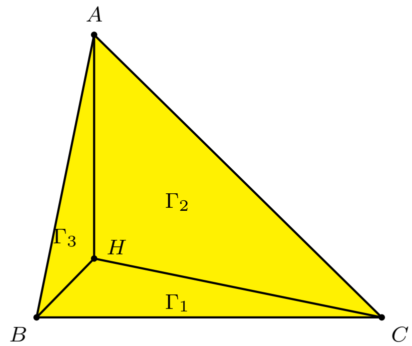
$$g := \gcd(a_1, a_2, a_3) = \gcd(n_1d_1, n_2d_2, n_3d_3).$$

The heights of Γ_0 are therefore

$$h'_i = \frac{2n_jd_jn_kd_k}{g_i \cdot g} = \frac{2}{g_i} \cdot \frac{n_jd_jn_kd_k}{\gcd(n_1d_1, n_2d_2, n_3d_3)}.$$

Note that $\frac{n_jd_jn_kd_k}{\gcd(n_1d_1,n_2d_2,n_3d_3)}$ is an *integer* prime to g_i. If h'_i is not an integer, then g_i must contain an odd prime divisor, by Proposition 3(4) again. □

9. Orthocentric Quadrangles and triples of simplifying factors



Triangle	Orthocenter	Half tangents	Simplifying factors
$\Gamma = ABC$	H	t_1, t_2, t_3	g_1, g_2, g_3(assumed odd)
$\Gamma_1 = HBC$	A	$\frac{1}{t_1}, \frac{1-t_3}{1+t_3}, \frac{1-t_2}{1+t_2}$	$2g_1, g_3, g_2$
$\Gamma_2 = AHC$	B	$\frac{1-t_3}{1+t_3}, \frac{1}{t_2}, \frac{1-t_1}{1+t_1}$	$g_3, 2g_2, g_1$
$\Gamma_3 = ABH$	C	$\frac{1-t_2}{1+t_2}, \frac{1-t_1}{1+t_1}, \frac{1}{t_3}$	$g_2, g_1, 2g_3$

Proposition 10. *The simplifying factors for the four (rational) triangles in an orthocentric quadrangle are of the form* (g_1, g_2, g_3), $(2g_1, g_2, g_3)$, $(g_1, 2g_2, g_3)$ *and* $(g_1, g_2, 2g_3)$, *with* g_1, g_2, g_3 *odd integers.*

Corollary 11. *Let* Γ *be a primitive Heron triangle. Denote by* Γ_i, $i = 1, 2, 3$, *the primitive Heron triangles in the similarity classes of the remaining three rational triangles in the orthocentric quadrangle containing* Γ. *The four triangles* Γ *and* Γ_i, $i = 1, 2, 3$, *are either all decomposable or all indecomposable.*

Example 4. The orthocentric quadrangle from Cheney's indecomposable $(25,34,39;420)$:

(a,b,c)	(t_1,t_2,t_3)	(g_1,g_2,g_3)
$(25,34,39;420)$	$\frac{5}{14},\ \frac{4}{7},\ \frac{6}{7}$	$(5,17,13)$
$(700,561,169;30030)$	$\frac{14}{5},\ \frac{3}{11},\ \frac{1}{13}$	$(10,17,13)$
$(855,952,169;62244)$	$\frac{9}{19},\ \frac{7}{4},\ \frac{1}{13}$	$(5,34,13)$
$(285,187,364;26334)$	$\frac{9}{19},\ \frac{3}{11},\ \frac{7}{6}$	$(5,17,26)$

Summary: To construct **indecomposable** primitive Heron triangles, one begins with a triple of **odd** integers (g_1, g_2, g_3), each greater than 1, such that

(i) the prime divisors of g_i, $i = 1, 2, 3$, are all congruent to 1 (mod 4),

(ii) $\gcd(g_1, g_2, g_3) = 1$.

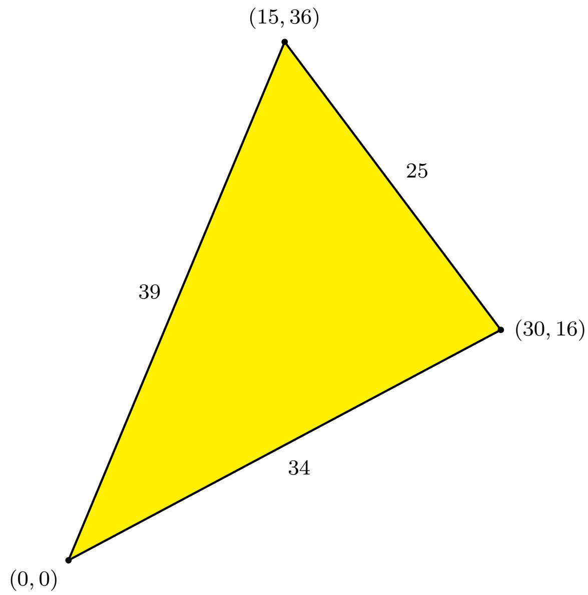
The primitive Heron triangle with $N(z_i) = g_j g_k$ is decomposable.

(g_1, g_2, g_3)	(d_1, n_1)	(d_2, n_2)	(d_3, n_3)	$(a, b, c; \triangle)$
$(5, 13, 17)$	$(14, 5)$	$(7, 6)$	$(7, 4)$	$(25, 39, 34; 420)$
	$(5, 14)$	$(9, 2)$	$(8, 1)$	$(175, 117, 68; 2520)$
	$(11, 10)$	$(7, 6)$	$(8, 1)$	$(275, 273, 68; 9240)$
	$(10, 11)$	$(9, 2)$	$(7, 4)$	$(275, 117, 238; 13860)$
$(5, 13, 29)$	$(4, 19)$	$(12, 1)$	$(8, 1)$	$(95, 39, 58; 456)$
	$(16, 11)$	$(8, 9)$	$(8, 1)$	$(110, 117, 29; 1584)$
	$(11, 16)$	$(12, 1)$	$(7, 4)$	$(220, 39, 203; 3696)$
	$(19, 4)$	$(8, 9)$	$(7, 4)$	$(95, 234, 203; 9576)$
$(5, 17, 29)$	$(22, 3)$	$(12, 1)$	$(2, 9)$	$(55, 34, 87; 396)$
	$(18, 13)$	$(9, 8)$	$(9, 2)$	$(65, 68, 29; 936)$
	$(18, 13)$	$(12, 1)$	$(6, 7)$	$(195, 34, 203; 3276)$
	$(22, 3)$	$(9, 8)$	$(7, 6)$	$(55, 204, 203; 5544)$
$(13, 17, 29)$	$(22, 3)$	$(16, 11)$	$(10, 11)$	$(39, 136, 145; 2640)$
	$(22, 3)$	$(19, 4)$	$(5, 14)$	$(429, 646, 1015; 87780)$
	$(18, 13)$	$(19, 4)$	$(11, 10)$	$(1521, 646, 1595; 489060)$
	$(18, 13)$	$(16, 11)$	$(14, 5)$	$(1521, 1496, 1015; 720720)$

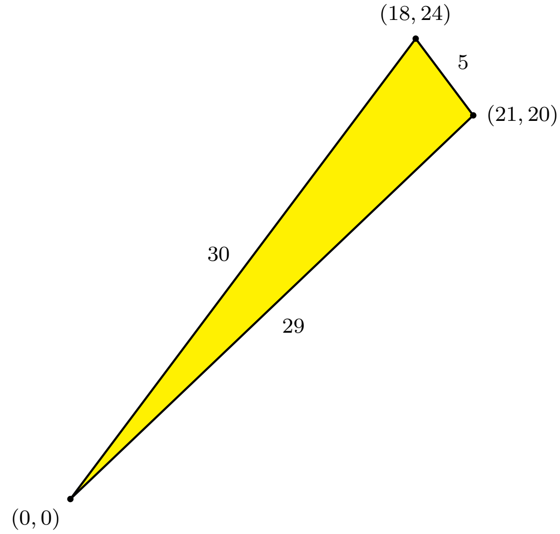
Further examples can be obtained by considering the orthocentric quadrangle of each of these triangles.

10. Examples of indecomposable Heron triangles as lattice triangle

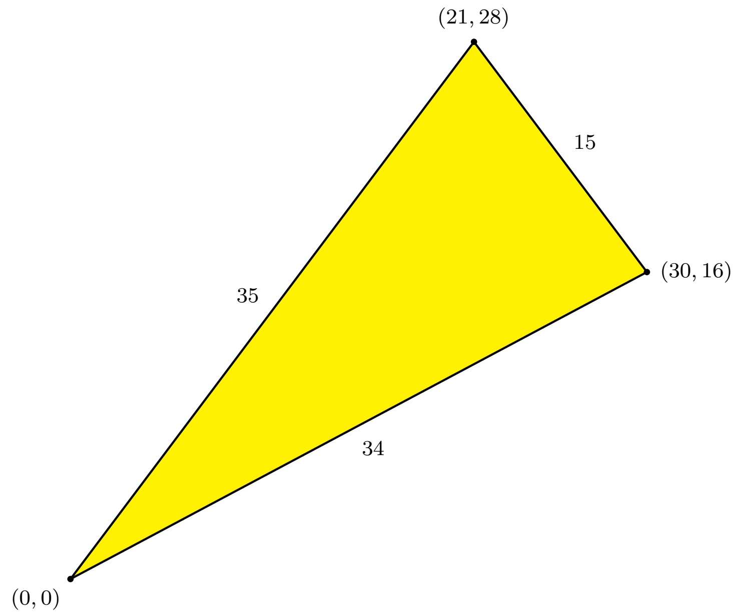
Theorem 12 (Y, MONTHLY, 108 (2001) 261–263)**.**
Every Heron triangle is a lattice triangle.



The indecomposable Heron triangle
$(25, 34, 39; 420)$ as a lattice triangle



The smallest indecomposable Heron triangle
$(5, 29, 30; 72)$ as a lattice triangle



The smallest acute indecomposable Heron triangle
$(15, 34, 35; 252)$ as a lattice triangle

THANK YOU