Midcircles and the Arbelos

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Abstract We investigate the use of midcircles to find results in the Arbelos.

1 Introduction

Recall that a midcircle of two given circles is a circle that swaps the two given circles by inversion. Midcircles are in the same pencil of circles as the given circles. We can distinguish three cases:

- The two given circles intersect: there are two midcircles with centers at the centers of similitude of the given circles.
- One given circle is in the interior of the other given circle. Then there is one midcircle with center of similitude at the internal center of similitude of the given circles.
- One given circle is in the exterior of the other given circle. Then there is one midcircle with center at the external center of similitude of the given circles.

Clearly the tangency cases can be seen as limit cases of the above.

2 The Arbelos

Now consider an arbelos, consisting of two interior semicircles $O_1(r_1)$ and $O_2(r_2)$ and an exterior semicircle $O(r) = O_0(r)$. Their points of tangency are $A$, $B$ and $C$ as indicated in figure 1. The incircle of the arbelos is $(O')$ - constructions can be found in [6, 7]. We denote by $M_0$ the external center of similitude of $(O_1)$ and $(O_2)$, and by $M_n$ the internal center of similitude of $(O)$ and $(O_n)$ ($n = 1, 2$). The midcircles are $M_0(C)$, $M_1(A)$ and $M_2(B)$. We consider also the Pappus chains $(P_{0,n})$, $(P_{1,n})$ and $(P_{2,n})$, which are not tangent to $(O)$, $(O_1)$ and $(O_2)$ respectively. In these chains $(P_{0,0}) = (P_{1,0}) = (P_{2,0}) = (O')$, and for any $n > 0$, $(P_{i,n})$ is tangent to $(P_{i,n-1})$ and $(P_{i,n+1})$ ($i = 0, 1, 2$). See figure 1.

The midcircles leave $(O')$ and its reflection through $AB$ invariant. The circles centered at $A$, $B$ and $C$ respectively and perpendicular to $(O')$ each invert two of the circles forming the arbelos to tangents to $(O')$ perpendicular to $AB$ and leave $(O')$ and its reflection through $AB$ invariant. These six circles are thus

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1 We adopt notations as used in [3]: By $(PQ)$ we denote the circle with diameter $PQ$, by $P(r)$ the circle with center $P$ and radius $r$, while $P(Q)$ is the circle with center $P$ through $Q$ and $(PQR)$ is the circle through $P$, $Q$ and $R$. The circle $(P)$ is the circle with center $P$, and radius clear from context.
members of a pencil, and \(O'\) lies on the radical axis of this pencil. This leads to interesting observations.

1. For \(i = 1, 2, 3\) inversion in the midcircle \((M_i)\) leaves \((P_{i,n})\) invariant. Consequently
   - the point of tangency of \((P_{i,n})\) and \((P_{i,n})\) lie on \((M_i)\) and their common tangent passes through \(M_i\);
   - when \((i, j, k)\) is a permutation of \((0, 1, 2)\), then the points of tangency of \((P_{i,n})\) with \((O_j)\) and \((O_k)\) are collinear with \(M_i\).

2. Inversion in \((M_i)\) swaps \((P_{j,n})\) and \((P_{k,n})\) for \((i, j, k)\) any permutation of \((0, 1, 2)\). Hence
   - \(M_i, P_{j,n}\) and \(P_{k,n}\) are collinear;
   - the points of tangency of \((P_{j,n})\) and \((P_{k,n})\) to \((O_i)\) are collinear with \(M_i\);
   - the points of tangency of \((P_{m,n})\) to \((O_m)\) for \(m = j, k\) are collinear with \(M_i\).

3. Let \(I_i\) be the circle that inverts \((O_j)\) and \((O_k)\) respectively to the two tangents \(\ell_1\) and \(\ell_2\) to \((O')\) perpendicular to \(AB\). The Pappus chain \((P_{i,n})\) is inverted to a chain \((Q_n)\) of congruent circles to \((O')\) tangent to \(\ell_1\) and \(\ell_2\) as well, with \((Q_0) = (O')\). Now consider the circle \(K_n\) through the points of tangency of \((P_{i,n})\) with \((O_j)\) and \((O_k)\) and perpendicular to \(I_i\). Then by inversion in \(I_i\) we see that \(K_n\) also passes through the points of tangency of \((Q_n)\) with \(\ell_1\) and \(\ell_2\). Consequently the center of \(K_n\) lies on the line through \(O'\) parallel to \(\ell_1\) and \(\ell_2\), i.e. the radical axis of the pencil of \(I_i\) and \((M_i)\). So by symmetry \(K_n\) passes through the points of tangency \((P_{i,n})\) with \((O_j)\) and \((O_k)\) for other permutations \((i', j', k')\) of \((0, 1, 2)\) as well. \(K_n\) thus pass through eight points of tangency, and are members of the same pencil.

4. With a similar reasoning the circle \(L_n\) tangent to \(P_{i,n}\) and \(P_{i,n+1}\) at their point of tangency as well as to \((Q_n)\) and \((Q_{n+1})\) at their point of tangency, belongs to the same pencil as \(K_n\). See figure 2.

The circles \(K_n\) and \(L_n\) make equal angles to the three arbelos semicircles \((O), (O_1)\) and \((O_2)\). In the following section we dive deeper into circles making equal angles to three given circles.

### 3 Locus-property of midcircles

**Lemma 1** The midcircle(s) of two given circles is (are) the locus of centers of inversion that map the given circles to two congruent circles.

**Proof:** Consider two given circles \(C_1 = N_1(R_1)\) en \(C_2 = N_2(R_2)\) and let \(d = N_1N_2\). Choose a rectangular coordinate system with origin at \(N_1\) and the (positive) \(x\)-axis along \(N_1N_2\) and let \(P(x, y)\) be a point of the locus. We have

\[
C_1 : \quad x^2 + y^2 - R_1^2 = 0
\]

2
\[ C_2 : (x - d)^2 + y^2 - R_2^2 = 0 \]

By taking the power of inversion \( \Phi = x^2 + y^2 - R_1^2 \) the circle \( C_1 \) is mapped to itself. Let \( I \) and \( J \) be the intersections of \( PN_2 \) with the circle \( C_2 \). We have
\[
\begin{align*}
PN_2^2 &= (d - x)^2 + y^2 \\
PI &= PN_2 - R_2 \\
PJ &= PN_2 + R_2
\end{align*}
\]

For the inverse points \( I' \) and \( J' \) of \( I \) and \( J \) this gives
\[
P I' = \Phi / PI \quad \text{and} \quad PJ' = \Phi / PJ
\]
and since \( P \) belongs to the locus
\[
P I' - PJ' = 2R_1
\]  
(1)

or
\[
P J' - PI' = 2R_1
\]  
(2)

Equation (1) leads to the following equation for the coordinates of \( P \)
\[
M_{\text{ext}} : (R_1 - R_2)(x^2 + y^2) - 2R_1 dx - R_1(R_2^2 - R_1 R_2 - d^2) = 0
\]

which represents a circle with center at the external center of similitude of the circles \( C_1 \) and \( C_2 \). Since \( M_{\text{ext}} = R_2 \cdot C_1 - R_1 \cdot C_2 \) this circle belongs to the pencil of circles generated by \( C_1 \) and \( C_2 \). This midcircle is called the external midcircle. Its radius is given by
\[
R_{\text{ext}}^2 = \frac{R_1 R_2 (d + R_1 - R_2)}{(R_1 - R_2)^2}.
\]

Similarly (2) leads to the following equation for the coordinates of \( P \)
\[
M_{\text{int}} : (R_1 + R_2)(x^2 + y^2) - 2R_1 dx - R_1(R_2^2 + R_1 R_2 - d^2) = 0
\]

which represents a circle with center at the internal center of similitude of the circles \( C_1 \) and \( C_2 \). Since \( M_{\text{int}} = R_2 \cdot C_1 + R_1 \cdot C_2 \) this circle also belongs to the pencil of circles generated by \( C_1 \) and \( C_2 \). This circle is called the internal midcircle. Its radius is given by
\[
R_{\text{int}}^2 = \frac{R_1 R_2 (d + R_1 + R_2)}{(R_1 - R_2)^2}.
\]

\[ \square \]

Remark As noted above, the midcircles do not appear both (as real circles) in all situations.

Corollary 1.1 The common points of 3 midcircles of 3 given circles taken by pairs are the centers of inversion that map the 3 given circles to 3 congruent circles.

Let \( M_{pq} \) be the midcircle(s) of the circles \( C_p \) and \( C_q \) and let \( P \) be the radical center of the circles \( C_1, C_2 \) and \( C_3 \). By Lemma 1 we have
\[
M_{pq} = R_q C_p + \epsilon_{pq} R_p C_q
\]
with \( \epsilon_{pq} = \pm 1 \). If we choose \( \epsilon_{12} \cdot \epsilon_{23} \cdot \epsilon_{31} = -1 \) the centers of \( M_{12}, M_{23} \) and \( M_{31} \) are collinear and since \( P \) has the same power with respect to these circles they form a pencil and their common points \( X \) and \( Y \) are the poles of inversion mapping the circles \( C_1, C_2 \) and \( C_3 \) into congruent circles.

The number of common points that are the poles of inversion mapping the circles \( C_1, C_2 \) and \( C_3 \) into a triple of congruent circles depends on the configuration of these circles.
• The maximal number is 8 and occurs when each pair of circles $C_p$ and $C_q$ has 2 distinct intersection points. Of these 8 points 2 points correspond to the 3 external midcircles while the other 3 pairs correspond to 1 of the 3 combinations of 1 external midcircle and 2 internal midcircles;

• The minimal number is 0 and occurs for instance when the circles belong to one pencil of circles without points of intersection.

**Corollary 1.2** The locus of the centers of the circles that intersect 3 given circles $C_1 = A(R_1), C_2 = B(R_2), C_3 = C(R_3)$ at equal angles are 0, 1, 2, 3 or 4 lines through the radical center $P$ of these 3 circles and perpendicular to either the line containing the external centers of similitude of each pair of these circles or one of the lines containing the external center of similitude of one pair and the internal centers of similitude of the other 2 pairs of circles.

**Proof** Consider a common point $X$ of 3 midcircles of these circles with collinear centers. The other common point $Y$ is the reflection of $X$ through the line containing these centers. Consider an inversion $\tau$ with pole $X$ that maps circle $C_3$ to itself. Circles $C_1$ and $C_2$ become $C_1' = A'(R_3)$ and $C_2 = B'(R_3)$. If $P'$ is the radical center of the circles $C_1$, $C_2'$ and $C_3'$ then every circle $C = P'(R)$ will intersect these 3 circles at equal angles. When we apply the inversion $\tau$ once again to the circles $C_1'$, $C_2'$, $C_3$ and $C$ we get the 3 original circles $C_1$, $C_2$, $C_3$ and a circle $C'$ and since an inversion preserves angles circle $C'$ will also intersect these original circles at equal angles.

The circles orthogonal to all circles $C'$ are mapped by $\tau$ to lines through $P'$. This means that the circles orthogonal to $C'$ all pass through the inversion pole $X$. By symmetry they also pass through $Y$, and thus form the pencil generated by the triple of midcircles we started with. The circles $C'$ form therefore a pencil as well, and their centers lie on $XY$ as $X$ and $Y$ are the limit-points of this pencil.

**Remark** Not every point on the line leads to a real circle, and not every real circle leads to real intersections and real angles.

As an example we consider the $A-$, $B-$ and $C-$Soddy circles of a triangle $ABC$. Recall that the $A-$Soddy circle of a triangle is the circle with center $A$ and radius $s-a$, where $s$ is the semiperimeter of $\triangle ABC$. The area enclosed in the interior of $\triangle ABC$ by the $A-$, $B-$ and $C-$Soddy circles form a skewed arbelos, as defined in [4]. The circles $F_\phi$ making equal angles to the $A-$, $B-$ and $C-$Soddy circles form a pencil, their centers lie on the Soddy line of $ABC$, while the only real line of three centers of midcircles is the tripolar of the Gergonne point $X_7$ (following numbering in [2, 1]).

The points $X$ and $Y$ in the proof of Corollary 1.2 are limit points of the pencil generated by $F_\phi$. These points are (in barycentric coordinates)

$$
(r + 4R) \cdot X_7 \pm \sqrt{(3)}s \cdot I = (r_A + \epsilon a \frac{\sqrt{3}}{2} : r_B + \epsilon b \frac{\sqrt{3}}{2} : r_C + \epsilon c \frac{\sqrt{3}}{2}),
$$

for $\epsilon = \pm 1$ and where $r$, $r_A$, $R$ are inradius, $A-$exradius and circumradius respectively and $I$ is the incenter. The Fletcher-point $X_{1323}$ is their midpoint. See figure 3.
4 \( \lambda \)-Archimedean circles

Recall that in the arbelos the twin circles of Archimedes have radius \( r_A = \frac{r_1 r_2}{2} \). Circles congruent to these twin circles with relevant additional properties in the arbelos are called Archimedean.

Now let the homothety \( h(A, \mu) \) map \( O \) and \( O_1 \) to \( O' \) and \( O_1' \). In [3] we have seen that the circle tangent to \( O' \) and \( O_1' \) and to the line through \( C \) perpendicular to \( AB \) is Archimedean for any \( \mu \) within obvious limitations.

On the other hand from this we can conclude that when we apply the homothety \( h(A, \lambda) \) to the line through \( C \) perpendicular to \( AB \), to find the line \( d \), then the circle tangent to \( d \), \( O \) and \( O' \) has radius \( \lambda r_A \). In fact these circles are described in a different way in [5]. We will call circles with radius \( \lambda r_A \) and with relevant additional properties \( \lambda \)-Archimedean.

We can find a family of \( \lambda \)-Archimedean circles in a way similar to Bankoff’s triplet circle. An inversion proof to show that Bankoff’s triplet circle is Archimedean uses the inversion in \( A(B) \), that maps \( O \) and \( O_1 \) to two parallel lines perpendicular to \( AB \), and \( (O_2) \) and the Pappus chain \( (P_{2,n}) \) to a chain of tangent circles enclosed by these two lines. The use of a homothety through \( A \) mapping Bankoff’s triplet circle \( (W_3) \) to its inversion image, shows that it is Archimedean. We can use this homothety as \( (W_3) \) circle is tangent to \( AB \). This we know because \( (W_3) \) is invariant under inversion in \( (M_0) \), and thus \( (W_3) \) and \( (M_0) \) intersect perpendicularly in \( C \). In the same way we find \( \lambda \)-Archimedean circles.

**Proposition 2** Let \( V_{2,n} \) be the point of tangency of \( (P_{2,n}) \) and \( (O_1) \). Similarly let \( V_{1,n} \) be the point of tangency of \( (P_{1,n}) \) and \( (O_2) \). The circle \( (CV_{1,n}V_{2,n}) \) is \( n+1 \)-Archimedean.

A special circle of this family is \( (L) = (CV_{1,1}V_{2,1}) \), which tangent to \( (O) \) and \( (O') \) at their point of tangency \( Z \), as can be easily seen from the figure after inversion. We will meet again this circle in the final section. See figure 4.

Let \( W_{1,n} \) be the point of tangency of \( (P_{0,n}) \) and \( (O_1) \). Similarly let \( W_{2,n} \) be the point of tangency of \( (P_{0,n}) \) and \( (O_2) \). The circles \( (CW_{1,n}W_{2,n}) \) are invariant under inversion through \( (M_0) \), hence are tangent to \( AB \). We may consider \( AB \) itself as preceding element of these circles, as we may consider \( (O) \) as \( (P_{0,-1}) \). Inversion through \( C \) maps \( (P_{0,n}) \) to a chain of tangent congruent circles tangent to two lines perpendicular to \( AB \), and maps the circles \( (CW_{1,n}W_{2,n}) \) to equidistant lines parallel to \( AB \) and including \( AB \). The diameters through \( C \) of \( (CW_{1,n}W_{2,n}) \) are thus, by inversion back of these equidistant lines, proportional to the harmonic sequence. See figure 5.

**Proposition 3** The circle \( (CW_{1,n}W_{2,n}) \) is \( \frac{1}{n+1} \)-Archimedean.

5 Inverting the arbelos to congruent circles

Let \( F_1 \) and \( F_2 \) be the intersection points of the midcircles \( (M_0), (M_1) \) and \( (M_2) \) of the arbelos. Inversion through \( F_1 \) maps the circles \( (O), (O_1) \) and \( (O_2) \) to three congruent and pairwise tangent circles \( (E_{1,0}), (E_{1,1}) \) and \( (E_{1,2}) \). Triangle \( E_{1,0}E_{1,1}E_{1,2} \) of course is equilateral, and stays homothetic independent of the power of inversion.
The inversion through $F_i$ maps $(M_0)$ to a straight line which we may consider as the midcircle of the two congruent circles $(E_{i,1})$ and $(E_{i,2})$. The center $M_0'$ of this degenerate midcircle we may consider at infinity. It follows that the line $F_iM_0 = F_iM_0'$ is parallel to the central $E_{i,1}E_{i,2}$ of these circles. Hence the lines through $F_i$ parallel to the sides of $E_{i,1}E_{i,2}E_{i,3}$ pass through the points $M_0$, $M_1$ and $M_2$.

Now note that $A$, $B$, and $C$ are mapped to the midpoints of triangle $E_{i,0}E_{i,1}E_{i,2}$, and the line $AB$ thus to the incircle of $E_{i,0}E_{i,1}E_{i,2}$. The point $F_i$ is thus on this circle, and from inscribed angles in this incircle we see that the directed angles $(F_iA, F_iB), (F_iB, F_iC), (F_iC, F_iA)$ are congruent modulo $\pi$.

**Proposition 4** The points $F_1$ and $F_2$ are the Fermat-Torricelli points of degenerate triangles $ABC$ and $M_0M_1M_2$.

Let the diameter of $(O')$ parallel $AB$ meet $(O')$ in $G_1$ and $G_2$ and Let $G'_1$ and $G'_2$ be their feet of the perpendicular altitudes on $AB$. From Pappus’ theorem we know that $G_1G_2G'_1G'_2$ is a square. Construction 4 in [6] tells us that $O'$ and its reflection through $AB$ can be found as the Kiepert centers of base angles $\pm \arctan 2$. Multiplying all distances to $AB$ by $\sqrt{3}$ implies that the points $F_i$ form equilateral triangles with $G'_1$ and $G'_2$. See figure 6.

A remarkable corollary of this and Proposition 4 is that the arbelos erected on $M_0M_1M_2$ shares its incircle with the original arbelos. See figure 7.

Let $F_1$ be at the same side of $ABC$ as the Arbelos semicircles. The inversion in $F_1(C)$ maps $(O), (O_1)$ and $(O_2)$ to three 2-Archimedean circles $(E_0)$, $(E_1)$ and $(E_2)$, which can be shown with calculations, that we omit here. The 2-Archimedean circle $(L)$ we met earlier meets $(E_1)$ and $(E_2)$ in their "highest" points $H_1$ and $H_2$ respectively. This leads to new Archimedean circles $(E_1H_1)$ and $(E_2H_2)$, which are tangent to Bankoff’s triplet circle. Note that the points $E_1, E_2, L$, the point of tangency of $(E_0)$ and $(E_1)$ and the point of tangency of $(E_0)$ and $(E_2)$ lie on the 2-Archimedean circle with center $C$ tangent to the common tangent of $(O_1)$ and $(O_2)$. See figure 8.

**References**


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