An Elementary Proof of the Erdős-Mordell Theorem

Inki Han

Abstract

We give an elementary proof of the Erdős-Mordell inequality using congruence of triangles, area of triangle, and the inequality $\frac{x}{y} + \frac{y}{x} \geq 2$ (for positive real numbers $x, y$).

The following inequality was conjectured by Erdős in [1]:

**Theorem.** If $O$ is a point in the interior of a triangle $ABC$ whose distances are $R_A, R_B, R_C$ from the vertices of the triangle and $K_a, K_b, K_c$ from its sides, then $2(K_a + K_b + K_c) \leq R_A + R_B + R_C$ with equality if and only if the triangle is equilateral and $O$ is its center.

Several proofs of this inequality have been given using trigonometry in [2], a theorem of Pappus in [3], a theorem of Ptolemy in [5, 7], angular computations with similar triangles in [4], and area inequality in [6].

The purpose of this work is to give another elementary proof. We use congruence of triangles, area of triangle, and the inequality $\frac{x}{y} + \frac{y}{x} \geq 2$ (for positive real numbers $x, y$) taught in secondary school.

![Figure 1](image)

Let $a, b,$ and $c$ be the lengths of the sides of the triangle $ABC$ opposite the respective vertices $A, B,$ and $C$. Consider points $M, N$ respectively lying on the sides $AB, AC$ satisfying $AM = AC = c, AN = AB = b$ (see figure 1). Then the triangles $ABC$ and $ANM$ are congruent, and $MN = BC = a$. In
order to prove the Erdős-Mordell theorem we consider two cases: (1) $O$ exists interior to intersection of the triangles $ABC$ and $AMN$ or its boundary; (2) $O$ exists interior to only triangle $ABC$ not to triangle $AMN$ or its boundary.

Case 1. $O$ exists interior to intersection of the triangles $ABC$ and $AMN$ or its boundary. We will prove the inequality $aR_A \geq bK_c + cK_b$ using triangle $AMN$. Let $K'_a, h'_A$ be the distance from $O, A$ to $MN$ respectively, and $S_{AMN}$ be the area of the triangle $AMN$ (see figure 2). Then,

$$2S_{AMN} = ah'_A, 2S_{AMN} = aK'_a + bK_c + cK_b$$

Figure 2

From these equalities we have

$$ah'_A = aK'_a + bK_c + cK_b, a(h'_A - K'_a) = bK_c + cK_b.$$ 

Since $h'_A - K'_a \leq R_A$ (in figure 2), the inequality $bK_c + cK_b \leq aR_A$ is proved. This inequality can be written as followings:

$$\frac{b}{a}K_c + \frac{c}{a}K_b \leq R_A.$$ 

Similarly, the following inequalities can be proved:

$$\frac{a}{b}K_c + \frac{c}{b}K_a \leq R_B, \frac{a}{c}K_b + \frac{b}{c}K_a \leq R_C.$$ 

Adding these inequalities, we have

$$\left(\frac{c}{b} + \frac{b}{c}\right)K_a + \left(\frac{c}{a} + \frac{a}{c}\right)K_b + \left(\frac{b}{a} + \frac{a}{b}\right)K_c \leq R_A + R_B + R_C.$$ 

From this and the inequality $\frac{x}{y} + \frac{y}{x} \geq 2$ for positive real numbers $x, y$ the inequality $2(K_a + K_b + K_c) \leq R_A + R_B + R_C$ is proved. It is easy to check that equality holds if and only if $AB = BC = CA$ and $O$ is its center.
Case 2. $O$ exists interior to only triangle $ABC$ not to triangle $AMN$ or its boundary. We will prove the inequality $aR_A \geq bK_c + cK_b$ using triangle $AMN$ (see figure 3). In figure 3 we have

$$2S_{AMN} = ah_A', 2S_{AMN} = bK_c + cK_b - aK_a'.$$

From these equalities, we have

$$ah_A' = bK_c + cK_b - aK_a', a(h_A' + K_a') = bK_c + cK_b.$$

Since $h_A' + K_a' \leq R_A$ (in figures 3), the inequality $bK_c + cK_b \leq aR_A$ is proved. Similarly, we can prove the inequality $2(K_a + K_b + K_c) \leq R_A + R_B + R_C$.

References


Inki Han: Department of Mathematics Education, Gyeongsang National University, Jinju, 660-701, Korea

E-mail address: inkiski@gsnu.ac.kr