Steiner’s Conic and Applications

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Abstract

We revisit Steiner’s construction of a conic, which asserts that the intersections of homologous rays through two fixed points with respect to a given projectivity lie on a conic. Moreover, we study an application of Steiner’s construction and we prove that the nine-point circle of a certain configuration can be regarded as a Steiner’s conic.

1 Introduction

Steiner’s construction of a conic states that the intersections of homologous rays through two fixed points with respect to a given projectivity on a line lie on a conic (see [5, p. 80] or [6, p. 82]). In the present note we give an alternative proof of Steiner’s construction and we extend it to a projectivity on a circle. Then we show how to use this theorem to prove a cyclicity property pointed out in Problem 10710 from American Mathematical Monthly, proposed by the second author in vol. 106, January 1999, p. 68. A first synthetic solution is due to A. Sinefakopoulos in American Mathematical Monthly, vol. 107, No. 6. (Jun–Jul, 2000), p. 572–573. Recently, this configuration has been studied also in [10] and it appears as an example in [3].

To further explore the interplay of Steiner’s conic with the underlying configuration, we conclude this paper with two additional solutions to the aforementioned cyclicity property: one by inversion and one by homothety. The solutions by inversion and by homothety explore how the geometry of transformations works on this particular configuration. The projective solution presented in section § 2 shows that the nine-point circle highlighted in the synthetic solutions can be regarded as a Steiner’s conic.

From the beginning, let us establish some common notation. We denote by $XY$ the line passing through the points $X$ and $Y$, by $[XY]$ the segment determined by $X$ and $Y$, by $|XY|$ the Euclidean distance of the segment $[XY]$, and by $XY$ the oriented segment (vector) defined by $X$ and $Y$. If $x$ and $y$ are the line coordinates of the points $X$ and $Y$, then the coordinate of $XY$ is $y - x$. The measure of an angle $\angle XYZ$ is denoted by $m(\angle XYZ)$. If two segments $[XY]$ and $[ZW]$ are congruent (equal in values), we write $[XY] \equiv [ZW]$.

2 Elements of Projective Geometry and Steiner’s Conic

In the beginning of this section we recall a few fundamental properties of the cross ratio, followed by some elements of real projective geometry. Let $A_1, A_2, A_3$ and $A_4$ be four distinct points, two by two, in
this order, on a line \( l \) in the Euclidean plane, and let \( x_1, x_2, x_3, \) and \( x_4 \) be their coordinates, respectively. We give following definition as it appears in, for example, [4, p. 77], [8, p. 248], [1, p. 161–164] and [7, p. 107]:

**Definition 1.** The cross ratio of the points \( A_1, A_2, A_3, A_4 \) on the line \( l \), denoted by \([A_1A_2A_3A_4]\), is the following real number:

\[
[A_1A_2A_3A_4] = \frac{\overrightarrow{A_1A_2}}{\overrightarrow{A_1A_4}} \div \frac{\overrightarrow{A_3A_2}}{\overrightarrow{A_3A_4}} = \frac{x_2 - x_1}{x_4 - x_1} \div \frac{x_2 - x_3}{x_4 - x_3},
\]

(1)

where \( \overrightarrow{A_iA_j} \) are oriented segments.

The cross ratio \([A_1A_2A_3A_4]\) is also known as the anharmonic ratio and the cross section of the points \( A, B, C, D \). Other notations for the cross ratio are also \((A_1, A_2, A_3, A_4)\) and \(\{A_1A_2, A_3A_4\}\). The notion of cross ratio can be extended to a pencil of four (concurrent) lines \(l_1, l_2, l_3\) and \(l_4\), as follows.

**Definition 2.** The cross ratio of a pencil of four lines \(l_1, l_2, l_3, l_4\) is defined to be the cross ratio of the four collinear points \(A_1, A_2, A_3, A_4\) obtained by intersecting of a generic secant line \(l\) with the pencil:

\([l_1l_2l_3l_4] = [A_1A_2A_3A_4]\).

The following result is normally attributed to Pappus. We prove it here for reasons of self-containment.

**Theorem 1.** The cross ratio of a pencil of four lines is well-defined, i.e. definition 2 is independent of the secant line \(l\).

**Proof.** Let \(O\) be the common intersection point of the lines \(l_1, l_2, l_3\) and \(l_4\). Let us denote by \(\alpha = \angle A_1OA_2\), \(\beta = \angle A_2OA_3\), and \(\gamma = \angle A_3OA_4\), the angles formed by the four lines of the given pencil. In order to compute the required cross ratio, we use the Law of Sines in the triangles \(\triangle OA_1A_3\), \(\triangle OA_2A_3\), \(\triangle OA_2A_4\), and \(\triangle OA_4A_1\), respectively:

\[
\frac{A_1A_2}{\sin \alpha} = \frac{OA_1}{\sin \angle OA_2A_1}, \\
\frac{A_1A_4}{\sin(\alpha + \beta + \gamma)} = \frac{OA_1}{\sin \angle OA_4A_1}, \\
\frac{A_3A_2}{\sin \beta} = \frac{OA_2}{\sin \angle OA_2A_3}, \\
\frac{A_4A_4}{\sin \gamma} = \frac{OA_3}{\sin \angle OA_4A_3}.
\]

But the angles \(\angle OA_2A_1\) and \(\angle OA_2A_3\) are supplementary angles, so their sines are equal, and the angles \(\angle OA_4A_3\) and \(\angle OA_4A_1\) are equal, therefore the cross ratio is:

\[
[l_1l_2l_3l_4] = \frac{\sin \alpha}{\sin(\alpha + \beta + \gamma)} \div \frac{\sin \beta}{\sin \gamma}.
\]

(2)

Therefore, the cross ratio depends only on the angles \(\alpha, \beta\) and \(\gamma\) of the given pencil and not on the secant line \(l\). \(\blacksquare\)
Remark 1. Similarly, it can be shown that the cross ratio of four collinear points in space is preserved by projections on any plane from an arbitrary exterior point.

The next step is to extend the notion of cross ratio to four points $A_1, A_2, A_3, A_4$ on a given circle $C$. For this, let $T$ be an arbitrary point on $C$, different than $A_1, A_2, A_3$ and $A_4$. Then the cross ratio defined by the four lines lines $TA_1, TA_2, TA_3, TA_4$ does not depend on $T$, because, in view of formula (2), it depends only on the angles formed by the pencil. Therefore, we give the following

Definition 3. The cross ratio of four points $A_1, A_2, A_3, A_4$ on a circle $C$ is defined to be the cross ratio of the pencil of lines $TA_1, TA_2, TA_3, TA_4$:

$$[A_1A_2A_3A_4]_C = [TA_1 TA_2 TA_3 TA_4],$$

where $T$ is an arbitrary point on the circle.

Remark 2. Using Remark 1 one can extend definition 3 to define projections on general conic sections, but this is a different topic and we will not pursue it in this paper.

A line segment range is defined to be a certain number of points on a line segment. For example, if the points $A_1, A_2, A_3, \ldots$ of coordinates $x_1 < x_2 < x_3 < \ldots$ lie on a line segment, then they are said to form a range, denoted by $\{A_1A_2A_3\ldots\}$. For example, it is easy to see that the range $\{A_1A_2A_3\}$ satisfies the relation of oriented segments

$$\vec{A_1A_2} + \vec{A_2A_3} + \vec{A_3A_1} = 0. \quad (3)$$

A perspectivity is defined to be a one-to-one correspondence between two ranges that are sections of a pencil of lines cut by two distinct secant lines. Finally, a projectivity is defined to be a combination of two or more perspectivities.

In the case of a pencil of four lines, a perspectivity is determined by the points of intersection of a secant line segment range and their images on another secant range, so it depends only on the (angles of) the pencil, and not on the ranges considered (via theorem 1). In this case, a projectivity, being a combination of perspectivities, can be defined directly as follows:

Definition 4. A projectivity on (the set of points of) a line $l$ is a map $\pi : l \rightarrow l$ such that, for any four points and their images, the cross ratios are preserved, i.e.

$$[A_1A_2A_3A_4] = [\pi(A_1) \pi(A_2) \pi(A_3) \pi(A_4)].$$

We call the couple $(A_i, \pi(A_i))$ a pair of homologous points of the projectivity $\pi$ on $l$, for all $1 \leq i \leq 4$.

Using theorem 1 again, one can prove the following well-known theorem (see [5, p. 34]).

Theorem 2 (Fundamental Theorem of Projective Geometry). A projectivity on a line $l$ is completely determined by three pairs of homologous points.

Remark 3. One can easily extend the definition of a projectivity on a line to a projectivity $\pi_C$ on a circle $C$, as follows. For four given points $A_i$, $1 \leq i \leq 4$, on a circle, we consider the pencil determined by the lines $TA_i$, for $T$ a fixed point on the circle, as in definition 3. Then we cut the pencil with a generic secant line $l$, and denote by $B_i$, $1 \leq i \leq 4$ the intersection points. Then there is a unique projectivity $\pi$ on the line $l$ corresponding to the points $B_i$ (theorem 1). Furthermore, we define $\pi_C(A_i)$ to be the intersection of the line $T\pi(B_i)$ with the circle, for $1 \leq i \leq 4$. Note that the projectivity $\pi$ does not depend on $l$ so we can choose the line $l$ generically, such that all intersections above exist.
The following is a characterization of pairs of homologous points (see also exercise 4 in [5, p. 118]).

**Proposition 1.** For any projectivity $\pi$ on a given line $l$, there exist four fixed real numbers $a, b, c$ and $d$, with $ad - bc \neq 0$, such that for any $x \in l$ its image $y = \pi(x)$ has the form:

$$y = \frac{ax + b}{cx + d} \quad (4)$$

Conversely, for any four arbitrary chosen real numbers $a, b, c$ and $d$, with $ad - bc \neq 0$, there exists a unique projectivity $y = \pi(x)$ on $l$ defined by (4).

**Proof.** Let $\pi$ be a projectivity on the line $l$, determined by three pairs of homologous points, say $(x_1, y_1), (x_2, y_2)$, and $(x_3, y_3)$. Let $(x, y)$ an arbitrary homologous pair on $l$. Then we have the relation:

$$\frac{y_2 - y_1}{y - y_1} \div \frac{y_2 - y_3}{y - y_3} = \frac{x_2 - x_1}{x - x_1} \div \frac{x_2 - x_3}{x - x_3}.$$ 

Solving for $y$ in terms of $x$, one finds an expression of the form (4), where $a, b, c$ and $d$ are uniquely determined in terms of $x_1, y_1, x_2, y_2, x_3, y_3$. One has to check that $a, b, c, d$ do not depend on the three chosen points. We leave the details to the reader.

Conversely, let $a, b, c$ and $d$ be four arbitrary chosen real numbers with $ad - bc \neq 0$. Let $x_i$ be the coordinates of four arbitrary chosen points on the given line $l$ and let $y_i = \frac{ax_i + b}{cx_i + d}$ on the line $l$, for all $1 \leq i \leq 4$. Then, for $i \neq j \in \{1, 2, 3, 4\}$ we have:

$$y_j - y_i = \frac{(ad - bc)(x_j - x_i)}{(cx_j + d)(ax_i + b)}.$$ 

Then an easy computation shows that the following cross ratios are equal:

$$\frac{y_2 - y_1}{y_4 - y_1} \div \frac{y_2 - y_3}{y_4 - y_3} = \frac{x_2 - x_1}{x_4 - x_1} \div \frac{x_2 - x_3}{x_4 - x_3}.$$ 

Therefore it follows that there is a projectivity $\pi$ which maps $x_i$ into $y_i$, for all $1 \leq i \leq 4$, uniquely determined by the (three out of) four given points.  

**Remark 4.** Using Remark 3, note that there is an obvious extension to Proposition 1 for a projectivity $\pi_C$ on a circle $C$. The relation (4) will take place on a line $l$ which cuts the pencil $TA_i$, for $1 \leq i \leq 4$.

**Definition 5.** Let $A$ and $B$ be two fixed points in the plane, and let $S$ be a line $l$ or a circle $C$, respectively, not passing through neither $A$ nor $B$. Let $\pi$ be a given projectivity on $S$, and let $(N, L)$ be a variable homologous pair. We call the lines $AN$ and $BL$ homologous rays with respect to the given data $\pi, A, B$ and $S$. Varying $N$ and $L$ on $S$, we obtain two families of homologous rays $\{AN\}, N \in S$ and $\{BL\}, L \in S$.

**Remark 5.** Note that, according to theorem 1, the definition above does not depend on $S$, but only on a pencil of four lines (two of them containing $N$ and $L$, respectively) for which $S$ is secant.

Using Proposition 1 and Remark 5 we prove below the following extension of Steiner’s construction of a conic. For an alternative approach, see e.g. [5, p. 80].
Theorem 3. The geometric locus of the intersection points of two homologous rays with respect to a given projectivity $\pi$ on a given line $l$ (or on a given circle $C$), and two fixed points $A$ and $B$, is a conic which passes through $A$ and $B$.

Proof. Without loss of generality, we can consider $A$ and $B$ of coordinates $(0,0)$ and $(1,0)$, respectively. For simplicity, we will prove the theorem only in the case of a projectivity on line; for a circle the proof can be reduced to a line using Remark 3. Again without loss of generality, we can consider the line $l$ to have equation $y = 1$, via Remark 5. Let $(N, L)$ be a variable homologous pair with respect to the given projectivity $\pi$ on $l$. Let $M$ be the intersection of $AN$ and $BL$. Moreover, denote by $(x_0, 1)$ the coordinates of $N$. It follows that $\left(\frac{ax_0 + b}{cx_0 + d}, 1\right)$ are the coordinates of $L$ (proposition 1). Then the coordinates of $M$, say $(x_M, y_M)$, satisfy the following equations:

$$
y_M = \frac{1}{x_0} x_M
$$

$$
y_M = \frac{cx_0 + d}{ax_0 + b - cx_0 - d}(x_M - 1).
$$

(5)

If we substitute $x_0 = \frac{x_M}{y_M}$ in the second equality, we obtain that the coordinates of $M$ satisfy the equation of the conic:

$$
cx^2 + (d - b)y^2 - (c + d - a)xy - cx - dy = 0.
$$

Note that $A(0,0)$ and $B(1,0)$ are also on this conic. Moreover, note that the conic will have different coefficients if one chooses a different line than $y = 1$, or a circle $C$ instead of a line.

Conversely, by reversing the arrows in the direct proof, for any point $M$ on the conic, its coordinates $(x_M, y_M)$ will be of the form (5), so related by a formula of type (4). This proves that $(x_M, y_M)$ is a homologous pair, via Proposition 1, which concludes our proof. $\blacksquare$

The reader should also notice that all results above can be further extended to projectivities on conics, via Remark 2, but this is a more general topic and we will not pursue it further in this paper.
3 An Application of Steiner’s Construction

First, let us recall the classical Euclidean geometry result which asserts that the midpoints of the sides of a triangle, the feet of altitudes and the midpoints of the segments determined by the vertices and the orthocenter are all concyclic, lying on the *nine-point circle*.

In what follows, we apply Steiner’s construction to prove the statement of the problem mentioned in the introduction. To better serve our exposition, we slightly rephrase the Problem 10710 from the *American Mathematical Monthly* in the following form:

**Problem 1.** Given a triangle $\Delta ABC$, let us denote by $D \in [BC]$, $E \in [AB]$, $F \in [AC]$ the contact points of its incircle with its sides, and let $I$ be the incenter. The parallel line through $A$ to $BC$ intersects $DE$ and $DF$ in $M$ and $N$, respectively. Let $T$ and $L$ be the midpoints of $MD$ and $ND$, respectively. Prove that the points $A, L, F, I, E$ and $T$ are concyclic.

**Solution.** First we notice that $\angle AME \equiv \angle BDE$ from the parallelism $AM \parallel BD$. Furthermore, $\angle DEB \equiv \angle AEM$ as opposite angles. Because $[BE] \equiv [BD]$, being tangents from the same point $B$ to the incircle, it follows that $\angle DEB \equiv \angle BDE$. Put together, these equality of angles lead to $\angle AME \equiv \angle AEM$, so $[AM] \equiv [AE]$. Similarly, we obtain $[AF] \equiv [AN]$. Since $[AE] \equiv [AF]$, being tangents from the same point $A$ to the incircle, we get $[AM] \equiv [AN]$, so $A$ is the midpoint of $MN$. Moreover, from $[AM] \equiv [AE] \equiv [AF] \equiv [AN]$, it follows that the triangles $\Delta MFN$ and $\Delta NEM$ are right triangles, with the right angles in $F$ and $E$, respectively. Therefore $MF$ and $NE$ are altitudes in $\Delta MDN$. Their intersection, denoted by $H$, is the orthocenter of $\Delta MDN$ and it lies on the incircle of $\Delta ABC$, i.e. $\angle HDF = 90^\circ$, and so $HD$ is a diameter of the incircle (see Figure 2).

Furthermore, in the triangles $\Delta HND$ and $\Delta HMD$ the segments $[LI]$ and $[TI]$ are midlines, respec-
tively, therefore $LI \perp MD$ and $TI \perp ND$. It follows that the angles \( \angle ATI \) and \( \angle ALI \) are right angles. This last fact motivates the following construction.

Consider the midpoint $U$ of the segment $[AI]$. Consider also $O_1$ on the perpendicular bisector of the line segment $[AI]$ such that $[O_1U] \equiv [AU] \equiv [UI]$, as in Figure 2. Observe that the triangle \( \triangle AO_1I \) is a right angle triangle, thus $m(\angle AO_1I) = 90^\circ$. Without loss of generality, we can assume that $O_1$ is in the same half-plane determined by the line $AB$ as $C$. Similarly one can get the same solution if $O_1$ is in the complementary half plane. Denote by $C_1$ the circle of center $O_1$ and radius $|O_1A| = |O_1I|$.

Consider $W_1$ and $W_2$, two mobile points on $C_1$ such that $m(\angle W_1O_1W_2) = 90^\circ$ and such that $W_2$ is $A$ when $W_1$ is $I$. Consider the rays $AW_2$ and $IW_1$. Note that the pair $(W_1, W_2)$ is a homologous pair for a given projectivity of the circle (see Remark 5), therefore the rays $AW_2$ and $IW_1$ are homologous. By Theorem 3, the point $P$ determined by $AW_2 \cap IW_1$ belongs to a Steiner’s conic.

But $m(\angle API)$ is half the sum of the arc measures $AI$ and $W_1W_2$, which is $90^\circ$. Thus, this conic is a circle $C$ with diameter $AI$. Because $\angle ATI$ and $\angle ALI$ are right angles, it follows that $L$ and $T$ are on this circle. Similarly for $E$ and $F$. The reader should immediately notice that the circle $C$ is the nine-point circle for the triangle $\triangle MDN$.

Let us further notice that, if we denote $\{W'_2\} := AL \cap C_1$, and $\{W_2\} := IL \cap C_1$, we have that $m(\angle W'_1W_2W'_2) = m(\angle ALI) = 90^\circ$. It means that $(W'_1W'_2)$ is a particular homologous pair with respect to our projectivity, i.e. $L$ lies on the Steiner’s conic (the circle) $C$. Similarly for $T$. Therefore the circle from our problem can be regarded as a particular Steiner’s conic determined by a projectivity on the circle $C_1$. ■

To further explore the role of Steiner’s conic with the underlying configuration, we conclude this paper with two additional solutions to problem 1, making use of geometric transformations: one by inversion and one by homothety.

Solution by inversion. Using the first part of the proof above, we have that $AT \parallel ND$, thus $[AT]$ is a midline in $\triangle MDN$. Denote by $\{T_\lambda\} = AT \cap EF$ and $\{I_\lambda\} = AI \cap EF$. Since $[AT]$ is midline in $\triangle MDN$, we have $\angle MTA \equiv \angle MDN$. Also, the measure of each of $\angle AEF$ and $\angle MDN$ is half the measure of the arc $EF$ of the incircle. The equality $\angle MDN \equiv \angle MTA \equiv \angle AEF$ implies that the
triangles $\Delta AET_A$ and $\Delta ATE$ are similar. Therefore $|AT| \cdot |AT_A| = |AE|^2$. Using the leg theorem in the right triangle $\Delta AEI$ we get $|AI_A| \cdot |AI| = |AE|^2$. We consider now the inversion of pole (center) $A$ and power $|AE|^2$. This inversion transforms the points $T$ and $I$ into the points $T_A$ and $I_A$, which lie on $EF$. Similarly, the inverse of $L$, denoted $L_A$ lies on the line $EF$. Since their images are collinear, the points $L, I$, and $T$ must lie on a circle $C$, which passes through $A$, the pole of inversion. Thus, the circle passing through the midpoints of the sides of $\Delta M DN$ is the nine-point circle in this particular triangle.

**Solution by homothety.** Recall that in the projective solution above we proved that the triangles $\Delta MF N$ and $\Delta N EM$ are right triangles, with the right angles in $F$ and $E$, respectively, and that $MF$ and $NE$ are altitudes in $\Delta NDM$. Their intersection $H$, is the orthocenter of $\Delta MND$ and lies on the incenter of $\Delta ABC$, and $2|HI| = |HD|$.

We recall here the synthetic geometry result which asserts that the symmetric of the orthocenter of a triangle with respect to the midpoint of a side lies on the circumcircle. Therefore, the symmetric points of $H$ with respect to the midpoints of the segments $[MN], [MD]$ and $[ND]$ lie on the circumcircle of $\Delta M DN$.

Consider now the homothety $\rho$ centered in $H$ and ratio $1/2$. Thus, the circumcircle of the triangle $\Delta M DN$ is transformed by $\rho$ into a circle passing through the midpoints of its sides. Since $|HI| = \frac{1}{2}|HD|$, we obtain that $I$ lies on the circle passing through $A, L$, and $T$, i.e. the nine-point circle for $\Delta M DN$.

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