Classical Subspaces of Symplectic Grassmannians

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1 Introduction and Basic Concepts

We assume the reader is familiar with the concepts of a partial linear rank two incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ (also called a point-line geometry) and the Lie incidence geometries. For the former we refer to articles in [B] and for the latter see the paper [Co].

The collinearity graph of $\Gamma$ is the graph $(\mathcal{P}, \Delta)$ where $\Delta$ consists of all pairs of points belonging to a common line. For a point $x \in \mathcal{P}$ we will denote by $\Delta(x)$ the collection of all points collinear with $x$. For points $x, y \in \mathcal{P}$ and a positive integer $t$ a path of length $t$ from $x$ to $y$ is a sequence $x_0 = x, x_1, \ldots, x_t = y$ such that $\{x_i, x_{i+1}\} \in \Delta$ for each $i = 0, 1, \ldots, t - 1$. The distance from $x$ to $y$, denoted by $d(x, y)$ is defined to be the length of a shortest path from $x$ to $y$ if some path exists and otherwise is $+\infty$.

By a subspace of $\Gamma$ we mean a subset $S$ such that if $l \in \mathcal{L}$ and $l \cap S$ contains at least two points, then $l \subseteq S$. $(\mathcal{P}, \mathcal{L})$ is said to be Gamma space if, for every $x \in \mathcal{P}, \{x\} \cup \Delta(x)$ is a subspace. A subspace $S$ is singular provided each pair of points in $S$ is collinear, that is, $S$ is a clique in the collinearity graph of $\Gamma$. For a Lie incidence geometry with respect to a “good node” every singular subspace, together with the lines it contains, is isomorphic to a projective space, see [Co]. Clearly the intersection of subspaces is a subspace and consequently it is natural to define the subspace generated by a subset $X$ of $\mathcal{P}$, $\langle X \rangle_\Gamma$, to be the intersection of all subspaces of $\Gamma$ which contain $X$. Note that if $(\mathcal{P}, \mathcal{L})$ is a Gamma space and $X$ is a clique then $\langle X \rangle_\Gamma$ will be a singular subspace.
1.1 The Grassmannian Geometries

Let $\mathbb{F}$ be a field. Let $V$ be a vector space of dimension $m$ over $\mathbb{F}$. For $1 \leq i \leq m-1$, let $L_i(V)$ be the collection of all $i$–dimensional subspaces of $V$. Now fix $j, 2 \leq j \leq m-2$ and set $\mathcal{P} = L_j(V)$.

For pairs $(C, A)$ of incident subspaces of $V$ with $\dim(A) = a, \dim(C) = c$ let $S(C, A)$ consist of all the $j$–subspaces $B$ of $V$ such that $A \subset B \subset C$.

Finally, let $\mathcal{L}$ consist of all the sets $S(C, A)$ where $\dim A = j-1, \dim C = j+1$. The rank two incidence geometry $(\mathcal{P}, \mathcal{L})$ is the incidence geometry of $j$–Grassmannians of $V$, denoted by $G_j(V)$. We also use the notation $G_{m,j}(\mathbb{F})$ for the isomorphism type of this geometry.

1.2 The Symplectic Grassmannians

Now let $W$ be a space of dimension $2n$ over the field $\mathbb{F}$, $f$ a non-degenerate alternating form on $W$ so that $(W, f)$ is a non-degenerate symplectic space. For $X \subset W$ let $X^\perp = \{w \in W : f(x, w) = 0, \forall x \in X\}$. Recall that a subspace $U$ of $W$ is totally isotropic if $U \subset U^\perp$.

For $1 \leq k \leq n$, let $\mathcal{I}_k$ consist of all totally isotropic $k$–dimensional subspaces of $W$. Fix $k$ with $2 \leq k \leq n-1$ and set $\mathcal{P} = \mathcal{I}_k$. For a pair of subspaces $C \subset D \subset C^\perp$ (so $C$ is totally isotropic) where $\dim C = c < k < d = D$ let $T(D, C)$ consist of all the $k$–dimensional totally isotropic subspaces $U$ such that $C \subset U \subset D$. When $c = k-1, d = k+1$ we set $l(D, C) = T(D, C)$ and set $L = \{l(D, C) : C \subset D, \dim C = k-1, \dim D = k+1\}$. In this way we obtain another rank 2 incidence geometry $G = (P, L)$ which we refer to as the symplectic $k$–Grassmannians of $W$. We denote the isomorphism type of this geometry by $C_{n,k}(\mathbb{F})$. Note that two totally isotropic $k$–subspaces are on a line if they span a totally isotropic $k+1$ dimensional totally isotropic subspace. We remark that the automorphism group of the geometry $(P, L)$ is isomorphic to $PSP_{2n}(\mathbb{F})$.

1.3 Grassmannian Subspaces of Symplectic Grassmannians

When $E \subset F \subset W$, $\dim E = e, \dim F = f$ satisfy $e < k-1, f > k+1$ with $E, F$ totally isotropic, the collection $T(E, F)$ is a subspace of $(P, L)$ and is isomorphic to an ordinary Grassmannian geometry $G_{f-e,k-e}(\mathbb{F})$. Such a subspace is called “parabolic” since the stabilizer in $\text{Aut}(P, L)$ is a parabolic subgroup of $\text{Aut}(P, L)$. It is natural to ask: Is every subspace of $C_{n,k}(\mathbb{F})$ which is isomorphic to some $G_{m,j}(\mathbb{F})$ parabolic?

Actually, this is not quite the case as the following example demonstrates:

Assume $\text{char}(\mathbb{F}) = 2$ and $n-k \geq 2$. Let $U$ be a totally isotropic subspace of dimension $k-1$. Then $T(U^\perp, U)$ is a subspace of $G$ and is isomorphic to a symplectic polar space of rank $n-k+1 \geq 3, C_{n,k+1,1}(\mathbb{F})$. Since the characteristic is two this is isomorphic to the orthogonal polar space, $B_{n-k+1,1}(\mathbb{F})$ (the space of singular points and totally singular lines in a non-singular orthogonal space on a vector space of...
dimension $2(n - k + 1) + 1$. In turn this contains subspaces which are isomorphic to the hyperbolic orthogonal space on a vector space of dimension six, $D_{3,1}(F)$. However, this is isomorphic to $G_{4,2}(F)$ via the Klein correspondence. As we shall show in our main theorem, apart from the parabolic subspaces, these are the only other examples of Grassmannians subspaces of a symplectic Grassmannian:

**Main Theorem:** Let $S$ be a subspace of $C_{n,k}(F)$, $S \cong G_{m,j}(F)$.

Then either $S$ is parabolic or else $\text{char}(F) = 2$, $(m, j) = (4, 2)$ and $S$ is a subspace of $T(U^\perp, U)$ for some totally isotropic subspace $U$, $\dim U = k - 1$. Moreover, if $Y$ is the subspace spanned by all the elements of $S$ then $\dim Y/U = 6$.

Before proceeding to the proofs we introduce some notation:

**Notation:** Since we will generate all kinds of subspaces, of $W$ the symplectic space, of the geometry $(P, L)$, etc. we need to distinguish between these. When $X$ is some collection of subspaces or vectors from $W$ we will denote the subspace of $W$ spanned by $X$ by $\langle X \rangle_F$. When $X$ is a subset of $P$ we will denote the subspace $(P, L)$ generated by $X$ by $\langle X \rangle_G$. And, when $X$ is a subset of $(P, L)$ we will denote the subspace of this geometry generated by $X$ by $\langle X \rangle_{\Gamma}$.

For a point $p \in P$ we will denote by $\Delta(x)$ the collection of all points of $P$ which are collinear with $x$ in $(P, L)$ (including $p$). For a point $p \in P$ we will use $\gamma(p)$ to indicate the points of $P$ where are collinear with $p$.

## 2 Properties of Grassmannians

In this short section we recall some properties of a Grassmannian incidence geometry $G_j(V) \cong G_{m,j}(F)$. We omit the proofs because they are either well known or entirely straightforward to prove.

**Lemma 2.1.** i) There are two classes of maximal singular subspaces of $(P, L)$ with representatives $S(V, D)$ where $\dim D = j - 1$ and $S(E, 0)$ where $\dim E = j + 1$. $S(V, D) \cong PG_{m-j}(F)$ and $S(E, 0) \cong PG_j(F)$. Those of the first class will be referred to as type one and the second class as type two.

ii) If $M_1$ and $M_2$ are maximal singular subspaces and $M_1 \cap M_2$ is a line then $M_1$ and $M_2$ are in different classes. If $M_1 \cap M_2$ is a point then they are in the same class.

**Lemma 2.2.** Let $M$ be a maximal singular subspace of $P$ of type one. Then $\langle M \rangle_{\Gamma} = V$.

Now let $U$ be a hyperplane of $V$ and $X$ a one space of $V, X$ not contained in $U$. Set $P(U) = \{x \in P : x \subset U\}$ and $P_X = \{x \in P : X \subset x\}$.

**Lemma 2.3.** i) $P(U)$ is a subspace of $P$ and $P(U) \cong G_{m-1,j}(F)$.

ii) $P_X$ is a subspace of $P$ and $P_X \cong G_{m-1,j-1}(F)$. 
Lemma 2.4. The diameter of the collinearity graph of $\mathcal{P}(V)$ is isomorphic to $\mathbb{P}G_{m-j-1}(F)$. Furthermore, $\langle x, \gamma(x) \cap \mathcal{P}(x) \rangle_{\mathbb{P}G_{m-j}(F)}$ is a maximal singular subspace of $\mathcal{P}$.

Lemma 3.1. i) The symplectic Grassmannian space $(P, L) \cong C_{n,k}(F)$ has two classes of maximal singular subspaces with representatives $T(B, 0)$ where $B$ is a totally isotropic subspace of $W$, $\dim B = k + 1$, and $T(C, A)$ where $A$ and $C$ are incident totally isotropic subspaces of $W$, where $\dim A = k - 1, \dim C = n$. In the former case $T(B, 0) \cong \mathbb{P}G_{k}(F)$ and in the latter $T(C, A) \cong \mathbb{P}G_{n-k}(F)$. We refer to the first as type one maximal singular subspaces and the latter as type two.

ii) If $M_1$ and $M_2$ are maximal singular subspaces of different types then either $M_1 \cap M_2$ is empty or a line.

iii) If $M_1$ and $M_2$ are type one maximal singular subspaces then $M_1 \cap M_2$ is either empty or a point.

Definition

A sym of $(P, L)$ is a maximal geodesically closed subspace which is isomorphic to a polar space.

Lemma 3.2. There are two classes of syms in $(P, L)$. One class has representative $T(E, D)$ where $D \subset E$ are totally isotropic subspaces, $\dim D = k - 2$, $\dim E = k + 2$. In this case $T(E, D) \cong D_{3,1}(F)$ the polar space of a non-degenerate six dimensional orthogonal space with maximal Witt index. The second class has representative $T(C^\perp, C)$ where $C$ is a totally isotropic subspace, $\dim C = k - 1$. In this case $T(C^\perp, C)$ is isomorphic to the polar space of a non-degenerate symplectic space of dimension $2(n - k + 1)$. We refer to the former as a type one sym and the latter as a type two sym.

Lemma 3.3. There are two classes of points at distance two in $G = (P, L)$. For one pair $(x, y)$ as subspaces of $W$, $\dim (x \cap y) = k - 2$ and $x \perp y$. The unique sym on $\{x, y\}$ is $T(x + y, x \cap y)$. For a representative $(x, y)$ of the second type, $\dim (x \cap y) = k - 1$ and $(x + y)/(x \cap y)$ is a non-degenerate two space. The unique sym on such a pair is $T((x \cap y)^\perp, x \cap y)$.
4 Proof of the Main Theorem

In this section we prove our main theorem. Our proof is by induction on \( N = n + k + m + \min\{j, m - j\} \).

Lemma 4.1. If \( S \cong \mathcal{G}_{4,2}(\mathbb{F}) \) then the main theorem holds.

**Proof**: Assume \( S \cong \mathcal{G}_{4,2}(\mathbb{F}) \). Since \( S \) is a polar space it is contained in some symp \( S \) of \((P, L)\). By Lemma (3.2) there are two possibilities for \( S \): either there are totally isotropic subspaces \( D \subset E, \dim D = k - 2, \dim E = k + 2 \) with \( S = T(E, D) \) or there is a totally isotropic subspace \( C, \dim C = k - 1 \) such that \( S = T(C^\perp, C) \). In the former case, since \( S \cong T(E, D) \) we get equality and the main theorem holds. In the latter case, let \( U = \langle S \rangle_\mathbb{F} \) a vector subspace of \( C^\perp \). The map taking \( x \in S \) to \( x/C \) is an embedding of the polar space \( S \) into \( \mathbb{P}G(U/C) \). Since \( S \) is strongly hyperbolic (see [CS]) it follows that \( \dim U/C = 6 \). Because we have an embedding from the orthogonal polar space \( S \) into the symplectic polar space \( T(C^\perp, C) \) it must also be the case that \( \text{char}(\mathbb{F}) = 2 \) which is one of the conclusions of the theorem.

Lemma 4.2. Assume that \( \min\{j, m - j\} = 2 \). Then the theorem holds.

**Proof**: Let \( S' \) be a subspace of \( S, S' \cong \mathcal{G}_{5,2}(\mathbb{F}) \) and let \( \mathcal{D} \) be a symp of \( S' \). Since \( \mathcal{D} \) is a polar space it is contained in a symp of \( G \). Suppose \( \mathcal{D} \) is contained in a type two symp \( T(C^\perp, C) \), \( C \) a totally isotropic subspace of \( W, \dim C = k - 1 \). Now for every point \( x \in S' \setminus \mathcal{D}, \Delta(x) \cap \mathcal{D} \) is a maximal singular subspace of \( \mathcal{D} \) (and isomorphic to \( \mathbb{P}G_2(\mathbb{F}) \)). The subspace \( \Delta(x) \cap \mathcal{D} \subset T(C^\perp, C) \) is a projective plane. Let \( M \) be a maximal singular subspace of \( T(C^\perp, C) \) containing \( \Delta(x) \cap \mathcal{D} \). Then it follows from Lemma (3.1) that \( M \) is a type two maximal singular subspace of \( G \). Now by Lemma (3.1) (ii), \( x \in M \subset T(C^\perp, C) \). Since the point \( x \in S' \setminus \mathcal{D} \) was arbitrary, it follows \( S' \subset T(C^\perp, C) \). However, since \( S' \) is not a polar space we have a contradiction.

As a consequence of this argument, all the symps of \( S \) are type one symps of \( G \). From this it follows that if \( x, y \in S, d(x, y) = 2 \) then as subspaces of \( W \) we have \( x \perp y \) by Lemma (3.3). Since the diameter of \( S \) is two by Lemma (2.4) it then follows that \( Y = \langle S \rangle_\mathbb{F} \) is a totally isotropic subspace of \( W \). Consequently, \( S \subset T(Y, 0) \). By Theorem (2.15) of [CKS] it follows that \( S \) is parabolic and the theorem holds.

The completion of the proof

It now follows that we may assume that \( m \geq 6 \) and \( \min\{j, m - j\} \geq 3 \). We continue with the notation of the introduction where \( V \) was introduced as an \( m \)-dimensional vector space and \((P, L)\) is the Grassmannian geometry of \( j \)-dimensional subspaces of \( V \). Let \( \tau : P \to S \) be an isomorphism of geometries. As in section two let \( U \) be a hyperplane of \( V \) and \( X \) a one-dimensional subspace of \( V \) such that \( X \) is not contained in \( U \) and set \( \mathcal{P}(U) = \{x \in \mathcal{P} : x \subset U\} \) and \( \mathcal{P}_X = \{x \in \mathcal{P} : X \subset x\} \). Also, set \( S_1 = \tau(\mathcal{P}(U)) \) and \( S_2 = \tau(\mathcal{P}_X) \).
Since $S_1 \cong G_{m-1,j}(\mathbb{F})$ and $(m - 1) + \min\{j, m - 1 - j\} < m + \min\{j, m - j\}$ it follows by our induction hypothesis that $S_1 = T(B_1, A_1)$ where $A_1 \subset B_1$ are totally isotropic subspaces with $\dim A_1 = a_1, \dim B_1 = b_1$ and $m - 1 = b_1 - a_1, j = k - a_1$.

Similarly, since $S_2 \cong G_{m-1,j-1}(\mathbb{F})$ and $(m - 1) + \min\{j - 1, (m - 1) - (j - 1)\} < m + \min\{j, m - j\}$ it follows that $S_2 = T(B_2, A_2)$ where $A_2 \subset B_2$ are totally isotropic subspaces with $\dim A_2 = a_2, \dim B_2 = b_2$ and $m - 1 = b_2 - a_2, j - 1 = k - a_2$.

Let $x \in S_1, y \in S_2, x, y$ collinear. Then by Lemma (2.3), $U_1 = \langle x, S_2 \cap \Delta(x) \rangle_G$ and $U_2 = \langle y, S_1 \cap \Delta(x) \rangle_G$ are maximal singular subspaces of $S$ which meet in a line.

Let $M_1$ be a maximal singular subspace of $P$ containing $U_i, i = 1, 2$. Then $M_1$ and $M_2$ come from different classes of $G$ by Lemma (3.1). Consequently, at least one of $M_1, M_2$ is of type 2. For the sake of argument, assume $M_1$ is of type 2. Then there is a maximal totally isotropic subspace $B$ and a $(k - 1)$-dimensional subspace $A \subset B$ such that $M_1 = T(B, A)$.

$M_1 \cap S_2 = T(B, A) \cap T(B_2, A_2) = S_2 \cap \Delta(x)$ is a maximal singular subspace of $S_2$.

It follows that $B_2 \subset B$ and that $M_1 \cap S_2 = T(B_2, A)$. Then $B_2 = \langle M_1 \cap S_2 \rangle_\mathbb{F} = \langle S_2 \cap \Delta(x) \rangle_\mathbb{F}$ by Lemma (2.2) which implies that $B_2 \subset x^\perp$ since $y' \in \Delta(x)$ implies $x \perp y'$.

Now assume that $x' \in S_1$ such that $x', x$ are collinear. Then by Lemma (2.3) it follows that $S_2 \cap \Delta(x')$ and $S_2 \cap \Delta(x')$ are in the same class of maximal singular subspaces of $S_2$. Therefore it also follows that $B_2 \subset (x')^\perp$.

Since the collinearity graph of $S_1$ is connected, it follows that for all $z \in S_1, B_2 \subset z^\perp$. Since $\langle S_1 \rangle_\mathbb{F} = B_1$ we have $B_1 \perp B_2$.

Set $D = B_1 + B_2$, a totally isotropic subspace. Now $S_1, S_2 \subset T(D, 0)$. Since $\langle S_1, S_2 \rangle_G = S$ (as follows from [BB], [CoSh], [RS]), it follows that $S \subset T(D, 0)$, for, if $x, y$ are collinear points of $P$ and $x, y \subset D$, then for every $z \in T(x + y, x \cap y)$ also $z \subset D$. Now we are done by Theorem (2.15) of [CKS].

References


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