and differentiate with respect to $a$. Equation (5.2) becomes

$$\cos \frac{\pi \theta}{2} \int_0^\infty e^{-xt} e^{-t} dt = \cos \frac{\pi \theta}{2} \frac{1}{(x + 1)^2}$$

$$= \cos \frac{\pi \theta}{2} \left[ 1 - 2x + 3x^2 - 4x^3 + \ldots \right].$$

With the cosine multipliers $1, 0, -1, 0, \ldots$ this series becomes

$$1 - 3x^2 + 5x^4 - 7x^6 + \ldots = \frac{1 - x^2}{(1 + x^2)^2} = \phi(x).$$

For this simple example the classical inversion is applicable:

$$\frac{1 - x^2}{(1 + x^2)^2} = \int_0^\infty t e^{-t \cos xt} dt.$$ 


References

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THE HEPTAGONAL TRIANGLE

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Three vertices of a regular heptagon can be connected to produce four distinct species of triangles, three of them isosceles and the fourth scalene. The latter, which we shall call “the Heptagonal Triangle”, is uniquely characterized by vertices whose angles, $A = \pi/7, B = 2\pi/7, C = 4\pi/7$, belong to a geometric progression with a common ratio of 2. A survey of the properties of this triangle will provide a better understanding of the regular polygon from which it is derived.

Since the earliest days of recorded mathematics, the regular heptagon has been virtually relegated to limbo. One could easily conjure up a variety of plausible reasons for this neglect. For example, unlike the equilateral triangle, the square and the regular pentagon, the regular heptagon cannot serve as a face of a regular polyhedron and as a result has had rather limited exposure to public view. Unlike the regular pentagon and the regular decagon, its properties, though striking, do not approach
those associated with the Divine Proportion, a ratio second only to $\pi$ in mathematical significance. Furthermore, unlike the equilateral triangle, the square and the regular hexagon, it has not enjoyed centuries of ornamental utility as a tiling for plane surfaces. To militate further against the acceptance of the heptagon into the community of familiar polygons, mathematicians have shown an aversion toward the study of the seven-sided figure perhaps because of their understandable inability to construct the figure with the well-known conventional Euclidean tools. But nonconstructibility is not necessarily synonymous with nonexistence, although many a frustrated scholar may have suspected this to be the case after a fruitless quest for heptagon material in the literature of mathematics.

The history of research on the regular heptagon from ancient times until the end of the nineteenth century could easily be encapsulated in one short paragraph. According to Arabian sources, Archimedes is believed to have written a book on the heptagon inscribed in a circle. If it is true that this work ever existed, it now seems to be irretrievably lost. Still, the question of its having been written appears credible because of a single surviving proposition, namely a “neusis” or “verging” construction of a regular heptagon. Archimedes accomplished this brilliant feat by using a marked instead of an unmarked ruler and by placing a certain line segment of definite length at a specially manipulated position in relation to certain other points and lines. Details elucidating this vague description may be found in Heath’s Manual of Greek Mathematics on pages 340–2 of the Dover reprint. The same source describes Heron’s approximate construction in which the apothem of an inscribed regular hexagon is considered to be almost equal to the side of a regular heptagon inscribed in the same circle. The apothem is equal to approximately 0.866026 times the side of the hexagon and would have to be stretched to only 0.867726 in order to qualify in a practical way as the side of the heptagon. Except for a downright silly and fallacious construction published by Thomas Hobbes, the eminent English philosopher, in his book entitled A Garden of Geometrical Roses, printed in London in 1727, the literature on heptagons is utterly barren. In 1796 the researches of the 19-year old Gauss inadvertently lent a hand in consigning the heptagon to oblivion simply because 7 happens to be a prime number that cannot be expressed in the form $2^n + 1$. In other words, the study of the regular heptagon was further discouraged by the belated proof of its non-constructibility with ruler and compasses.

In 1913 the late Victor Thébault of Tennie, France, directed his attention to an investigation of the long dormant heptagon and succeeded in bringing to light many surprising properties of great esthetic interest. He was attracted to this venture by the example set by Morley’s theorem, a beautiful proposition that arrived rather late on the geometrical scene probably because of the unconscious taboo associated with the forbidden angle trisection. (Morley’s theorem states that the intersections of the adjacent internal or external trisectors of a triangle are vertices of an equilateral triangle.) The purpose of this paper is to assemble a number of Thébault’s more interesting discoveries and to make available in English the essence of material hitherto published only in French. A further purpose is to offer some original
theorems and to review several heptagon problems, particularly those involving theorems previously published without proof in various editorial notes.

We start with a problem that exhibits a startling connection between the regular heptagon and the square inscribed in the same circle. (See the American Mathematical Monthly, problem E 1154, 1955, 584). While it shows how the side of an inscribed square can be precisely derived from elements of an inscribed regular heptagon, it offers no encouragement to wishful mystics who may still be yearning for the feasibility of the reverse procedure. The proposal offered by Victor Thébault was as follows:

The distance from the midpoint of side $AB$ of a regular convex heptagon $ABCDEFG$ inscribed in a circle to the midpoint of the radius perpendicular to $BC$ and cutting this side, is equal to half the side of a square inscribed in the circle. (Figure 1.)

Two solutions were published, one invoking the cosine law and the other using complex numbers. In the first solution, let $d$ denote the required distance, $R$ the circumradius and $\theta = \pi/7$. By the cosine law, we have

$$d^2 = R^2(1/4 + \cos^2 \theta - \cos \theta \cos 2\theta).$$

Since $\sin 3\theta = \sin 4\theta$, it follows that

$$3 \sin \theta - 4 \sin^3 \theta = 4 \sin \theta \cos \theta \cos 2\theta,$$

which reduces to

\[ (*) \quad \cos^2 \theta - \cos \theta \cos 2\theta = 1/4. \]

Hence $d = R\sqrt{2}/2$. 

![Figure 1](image-url)
(Note. The relation indicated by (*) applies uniquely to the regular heptagon and will be used later in this paper in the development of other heptagon properties. Indeed, it holds if $\theta$ is replaced by $n\theta$, where $n$ is any integer or zero.)

The second published solution, offered by Hüseyin Demir, designates the vertices $F, G, A, \ldots, E$ as the affixes of the 7th roots $1, e, e^2, \ldots, e^6$ of unity. Then the midpoints $U, V, \ldots$, of $AB$ and the concerned radius correspond to $u = (e^2 + e^3)/2$ and $v = -1/2$, whence

$$UV^2 = (u - v)(\bar{u} - \bar{v}) = (1 + e^2 + e^3)(1 + e^4 + e^5)/4$$

$$= (2 + 1 + e + \cdots + e^6)/4 = 1/2,$$

thus establishing the proposition.

Extending this method to diagonals, Demir states the following: The midpoints of the sides of the hexagon $ABGDCEA$ are equidistant from the point $V$, the common distance being half the side of the inscribed square. This means that the circle of radius $UV$, centered at $V$, bisects the segments $AB, BG, GD, DC, CE,$ and $EA$. Since the midpoints $W, X$ of $BG$ and $AE$ are symmetrical to those of $CE$ and $DG$ with respect to the diameter through $V$, the proof of this corollary may be abbreviated by showing that $VW = VX = UV$. The method of the first solution is also applicable to this proof.

Thébault mentioned the following additional properties of the regular heptagon $ABCDEFG$. Let $O$ be the center of the heptagon, $W$ the midpoint of $OF$, $M$ the point diametrically opposite $F$, $U$ the midpoint of $AB$, $V$ the midpoint of $OM$, and $J$ the point on $UB$ produced such that $UJ = UM$ (Figure 2). Then:
(1) $UW$ is equal to the diagonal of the square constructed on an apothem of the heptagon as a side.

(2) $OJ$ is equal to the diagonal of the square constructed on half the side of the inscribed equilateral triangle.

(3) $UV$ is tangent to the circle through $U$, $O$, $W$.

The proofs of (1) and (2) are easily obtained by the use of the identity (*). The relation $VO \cdot VW = UV^2$ proves (3).

The impact created by these astonishing relationships is evidenced by the editorial comment: "One cannot help but wonder if these properties, and the property of the problem, are just remarkable geometric accidents, or whether they are special cases of more general theorems involving, perhaps, other regular polygons. Karst did some work on the regular inscribed nonagon $ABCDEFGHI$. Here, if $U$ is the midpoint of side $AB$ and $V$ is the midpoint of the radius perpendicular to and cutting side $BC$, it can be shown that angle $OUV = 30^\circ$.

In response to an inquiry about the origin of these unusual discoveries, a letter from Thébault dated January 3, 1956 offered the following explanation and incidentally revealed two additional relationships resembling the one in problem E 1154:

"Let $ABC$ be the triangle in which $A = \pi/7$, $B = 2\pi/7$, and $C = 4\pi/7$. Let $M$ be the point of intersection of the internal bisector of $A$ with the circumcircle $(O)R$; let $N$ be the point of intersection of $(O)R$ with the internal bisector of $B$. The triangle $MCN$ is curious—also the quadrangle $OMCN$. For the latter—the squares of the bimedians (lines joining the midpoints of the opposite sides) have as expressions: $R^2/2$ (E 1154), $MN^2/2$, $MC^2/2$. That is the origin of E 1154…" (Figure 3).

To assist those seeking to verify the relationships mentioned by Thébault, we note that one of the bimedians connects the midpoints of the diagonals $MN$.
and CO. The proofs make generous use of the formula (*) in addition to a judicious manipulation of the sine and cosine sum-to-product identities.

**Additional properties of the heptagonal triangle.**

1. *The sum of the squares of the sides of the heptagonal triangle is equal to $7R^2$, where $R$ is the circumradius of the triangle.*

Applying the sine law to the sides $a, b, c$ opposite the angles $A = \pi/7, B = 2\pi/7, C = 4\pi/7$, and converting the resulting sine ratios to the corresponding cosines, we obtain

\[
(*) \quad \cos A = b/2a \quad \cos B = c/2b \quad \cos C = -a/2c,
\]

so that $\cos A \cos B \cos C = -1/8$. Then

\[
a^2 + b^2 + c^2 = 4R^2(\sin^2 A + \sin^2 B + \sin^2 C) \\
= 4R^2(2 + 2 \cos A \cos B \cos C) \\
= 4R^2(7/4) = 7R^2.
\]

Other methods of solution may be found in the February 1957 issue of the *American Mathematical Monthly*, pp. 110–112, problem E 1222.

2. *If $A', B', C'$ denote the feet of the altitudes from $A, B, C$, the orthic triangle $A'B'C'$ is similar to triangle $ABC$ and each side of the former is half the length of the corresponding side of the latter. (Figure 4.)*
In the cyclic quadrilateral $CB'BC'$, angle $A'C'B'/2 = \angle CBB' = 90^\circ - A - B = \pi/14 = A/2$. Similarly, angle $A'B'C' = C$ and angle $B'A'C' = B$. Thus the triangles are similar. Observe that the circumcircle of triangle $A'B'C'$ is the nine-point circle of triangle $ABC$. Since the radius of the nine-point circle is half that of the circumcircle, all the linear elements of the orthic triangle are half those of the corresponding elements of the parent triangle.

The heptagonal triangle is the only obtuse triangle displaying orthic similarity. As for acute triangles, the equilateral triangle is the only one similar to its orthic triangle. Here again, strangely enough, the ratio of similitude is 2. A problem relating to orthic similarity is number 681, on page 219 of the September 1968 issue of this Magazine.

3. If $a$, $b$, $c$ are the sides of the heptagonal triangle $ABC$ in which $C = 2B = 4A$, the side $a$ is half the harmonic mean of the other two sides.

Since $A = \pi/7$, it follows that $\sin 3A = \sin 4A$. Then

$$\sin A = \frac{\sin 2A}{2 \cos A} = \frac{\sin 2A \sin 4A}{2 \cos A \sin 3A} = \frac{\sin 2A \sin 4A}{\sin 2A + \sin 4A}.$$ 

With $a/\sin A = b/\sin B = c/\sin C = 2R$, we obtain $a = bc/(b + c)$, the required result. Equivalent expressions for $b$ and $c$ are $b = ac/(c - a)$ and $c = ab/(b - a)$.

See problem 189 in the Spring 1968 issue of the Pi Mu Epsilon Journal for two other treatments of the solution.

4. If $h_a$, $h_b$, $h_c$ are the altitudes to the sides $a$, $b$, $c$ of the heptagonal triangle $ABC$, then $h_a = h_b + h_c$.

Expressing the results of the preceding property in the form $1/a = 1/b + 1/c$ and using the relations $h_a = 2S/a$, $h_b = 2S/b$, $h_c = 2S/c$, where $S$ is the area of triangle $ABC$, we obtain the result $h_a = h_b + h_c$.

5. The following list of fundamental properties of the heptagonal triangle will be useful in deriving others. In each case, $A = \pi/7$, $B = 2\pi/7$ and $C = 4\pi/7$.

$$\sin A \sin B \sin C = \sqrt{7}/8.$$ 

$$\sin^2 A + \sin^2 B + \sin^2 C = 7/4.$$ 

$$\sin 2A + \sin 2B + \sin 2C = \sqrt{7}/2.$$ 

$$\sin^2 A \sin^2 B \sin^2 C = 7/64.$$ 

$$\sin^2 A \sin^2 B + \sin^2 A \sin^2 B + \sin^2 B \sin^2 C = 7/8.$$ 

$$\cos A \cos B \cos C = -1/8.$$ 

$$\cos^2 A + \cos^2 B + \cos^2 C = 5/4.$$ 

$$\cos^2 A \cos^2 B + \cos^2 A \cos^2 C + \cos^2 B \cos^2 C = 3/8.$$
\[
\cos 2A + \cos 2B + \cos 2C = -1/2 \\
\sin A + \sin B + \sin C = \sqrt{(7/2)} \\
\tan A \tan B \tan C = -\sqrt{7}. \\
\cot A + \cot B + \cot C = \sqrt{7}. \\
csc^2 A + csc^2 B + csc^2 C = 8. \\
sec^2 A + sec^2 B + sec^2 C = 24. \\
cot^2 A + cot^2 B + cot^2 C = 5. \\
\tan^2 A + \tan^2 B + \tan^2 C = 21. \\
sec^4 A + sec^4 B + sec^4 C = 416. \\
\cos^4 A + \cos^4 B + \cos^4 C = 13/16. \\
\sin^4 A + \sin^4 B + \sin^4 C = 21/16. \\
csc^4 A + csc^4 B + csc^4 C = 32. \\
sec 2A + sec 2B + sec 2C = -4.
\]

To obtain the above relations various methods are available. For example, one can start with the expansion of \(\sin 7x\) in terms of powers of \(\sin x\) to obtain the equation

\[64x^7 - 112x^5 + 56x^3 - 7x = 0,\]

the roots of which are \(0, \pm \sin \pi/7, \pm \sin 2\pi/7, \pm \sin 4\pi/7\). Then the fractions \(7/4, 7/8, 7/64\) represent the sums of \(\sin^2 A\), \(\sin^2 B\) and \(\sin^2 C\) taken one, two and three at a time.

Similar procedures combined with the use of well-known standard trigonometric identities lead to expressions involving the other trigonometric functions.

6. \textit{The sum of the squares of the altitudes of the heptagonal triangle is equal to half the sum of the squares of the sides of the triangle.}

To prove that \(h_a^2 + h_b^2 + h_c^2 = (a^2 + b^2 + c^2)/2\) convert the altitudes and the sides to their trigonometric equivalents in terms of sines and substitute the numerical values already obtained for \(\Sigma \sin^2 A \sin^2 B\) and for \(\Sigma \sin^2 A\).

7. \textit{The cotangent of the Brocard angle \(V\) of the heptagonal triangle is equal to \(\sqrt{7}\).}

Here again we make use of previously derived results. If \(S\) denotes the area and \(V\) the Brocard angle of the heptagonal triangle \(ABC\), \(\cot V = \cot A + \cot B + \cot C = (a^2 + b^2 + c^2)/4S\). The terms of this identity correspond to their numerical equivalents given in paragraph 5.

8. \textit{If} \(a, b, c\) \textit{are the sides of the heptagonal triangle} \(ABC\), \textit{then} \(b^2 - a^2 = ac\), \(c^2 - b^2 = ab\), and \(c^2 - a^2 = bc\).
Applying the relation \( a = c \cos B + b \cos C \) to the values of \( \cos B \) and \( \cos C \) used in (**) of paragraph 1, we obtain \( a = (c^2 + ab - b^2)/2b \), which reduces to \( ab = c^2 - b^2 \). Similarly, \( b^2 - a^2 = ac \) and \( c^2 - a^2 = bc \).

Combining these results and the relation \( \sin^2 x = 1 - \cos^2 x \) with the values of the cosines in terms of the sides given in property (**), we obtain \( \sin^2 A = (3a - c)/4a \), \( \sin^2 B = (3b - a)/4b \) and \( \sin^2 C = (3c - b)/4c \).

9. In the heptagonal triangle \( ABC \) whose sides are \( a, b, c \), we have

\[
\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 5.
\]

This follows immediately from (**), where \( b/a = 2 \cos A \), \( c/b = 2 \cos B \) and \( a/c = -2 \cos C \).

10. If \( a, b, c \) are the sides of the heptagonal triangle, then

\[
\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{2}{R^2}.
\]

This is a direct consequence of the relation \( \csc^2 A + \csc^2 B + \csc^2 C = 8 \), a property listed in paragraph 5.

11. If \( A', B', C' \) are the feet of the altitudes issuing from the vertices \( A, B, C \) of the heptagonal triangle, then \( BA' \cdot A'C = ac/4 \), \( CB' \cdot B'A = ab/4 \), and \( AC' \cdot C'B = bc/4 \).

This property follows from (**) of paragraph 1.

12. The exradius \( r_a \) relative to the vertex \( A \) of the heptagonal triangle \( ABC \) is equal to the radius of the nine-point circle of triangle \( ABC \). (Figure 5).

Let \( M, N, P \) denote the contacts of the excircle \((I_a)r_a \) with the sides \( AC, CB, AB \) respectively. The sides of the triangle \( MNP \) are parallel to those of the orthic triangle \( A'B'C' \) because their corresponding sides are perpendicular to the same angle bisectors of the heptagonal triangle \( ABC \). Hence the triangles \( ABC, A'B'C' \) and \( MNP \) are similar.

The line \( NP \) meets \( AC \) in \( Q \) and the triangles \( MAP, MQP, QAP \) and \( MQN \) are isosceles. It follows that

\[
AP = AM = AQ + QM = QP + QM = QN + NP + QM = MN + NP + PM
\]

and the triangles \( A'B'C', MNP \), having the same perimeter, are equal; their circumcircles have the same radius.

The Euler relation then gives

\[
OI_a^2 = R^2 + 2Rr_a = 2R^2,
\]

\( R \) being the radius of the circumcircle of triangle \( ABC \). Thus \( I_a \) is situated on the orthoptic circle of the circle \((O)\), that is, the circle concentric with \((O)\) and with a radius equal to \( R\sqrt{2} \).
The equality of the triangles $A'B'C'$ and $MNP$ permits us to say that the lines $NC'$, $PB'$ and $MA'$ are tangent to the nine-point circle and that the quadrilaterals $MA'C'P$, $MB'C'N$ and $PB'A'N$ are parallelograms.

13. **The internal angle bisectors of the angles $C$ and $B$ are equal respectively to the difference of the two adjacent sides; the external angle bisector of $A$ is equal to the sum of the adjacent sides.**

From $B$ draw a perpendicular to the internal bisector of angle $A$, cutting $BC$ in $K$; the triangles $KAB$, $KCB$ and $KFB$ are isosceles, where $F$ is the foot of the internal bisector of angle $B$. We then find that

$$AB = AK = AF + FK = BF + KB = BF + BC,$$

whence $BF = AB - BC$. 
Similarly, we have $CG = CA - BC$, where $CG$ is the internal bisector of angle $C$; also, $AL = CA + AB$, $AL$ being the external bisector of angle $A$.

14. The triangle formed by joining the feet of the internal angle bisectors of the heptagonal triangle $ABC$ is isosceles. (Figure 6.)

Let $D$, $E$, $F$ denote the feet of the interval angle bisectors issuing from $A$, $B$, $C$. We combine the relations $c = ab/(b-a)$ and $ab = c^2 - b^2$ of sections 3 and 8 respectively to yield $b/(a + c) = c/(b + c)$. Now $EC = ab/(a + c)$ and $BD = ac/(b + c)$. Hence $EC = BD$. Also, $FB = FC$, since the base angles $FBC$ and $FCB$ are each equal to $2\pi/7$. With the equality of angles $FCE$ and $FBC$, the triangles $FCE$ and $FBC$ are congruent and $FE = FD$.

15. The orthic triangle $A'B'C'$ and the medial triangle $M_1M_2M_3$ are congruent and in perspective. (Figure 7.)

The orthic similarity of the heptagonal triangle (with the ratio of similitude equal to 2) establishes the congruency of triangles $A'B'C'$ and $M_1M_2M_3$. Since the vertices of the two triangles are six points of the nine-point circle, the parallelism of the lines $C'M_1$, $M_3B'$ and $M_2A'$ is easily established by noting the equality of the angles $C'M_1B$, $M_3M_2B'$ and $(M_3B', A'B)$. An interesting sidelight is the inverse similarity of the triangles $A'C'M_1$ and $A'B'C'$.

16. The triangle $II_bI_c$ formed by the incenter of the heptagonal triangle and the excenters relative to $B$ and $C$ is similar to the triangle $ABC$, to its orthic triangle and to the pedal triangle of the nine-point center of triangle $ABC$. (Figure 8.)

The proof follows easily from the comparison of angles in the cyclic quadrilaterals $I_bAIC$ and $I_cBIA$.

17. The properties in this section are stated without proof. Most of the derivations are not difficult.
(a) The first Brocard point of the heptagonal triangle is the center of the nine-point circle and the second Brocard point lies on this circle.

(b) The segment of the Euler line contained between the circumcenter and the orthocenter is equal to the diagonal of the square constructed on the radius of the circumcircle. Stated differently, $OH = \frac{R}{\sqrt{2}}$.

(c) The segment connecting the incenter and the orthocenter of the heptagonal triangle is measured by the relation $IH^2 = \frac{R^2 + 4r^2}{2}$.

(d) The two tangents from the orthocenter to the circumcircle of the heptagonal triangle are mutually perpendicular.

(e) The center of the circumcircle of the tangential triangle coincides with the symmetric of the point $O$ with respect to $H$.

(f) The altitude from $B$ is half the length of the internal bisector of angle $A$.

The properties we have considered do not by any means exhaust the curiosities associated with the heptagonal triangle. The references listed at the end of this paper should be of assistance to any reader able and willing to explore the subject further in French journals.

Following the example set by Heron in offering an approximation to the side of a regular heptagon by using the apothem of a regular hexagon, we add a few oddities of our own:

(a) If $I$ is the incenter of the heptagonal triangle $ABC$, $BI$ is a good approximation for the side of a regular enneagon inscribed in the same circle.

(b) One-third of the length of the median to side $BC$ is a good approximation for the side of a regular hendecagon inscribed in the same circle.

(c) Denoting the centroid of the heptagonal triangle by $G$ and the circumcenter by $O$, the segment $OG$ is a good approximation for the side of a regular triskadecagon inscribed in the same circle.

We conclude with the observation that the heptagonal triangle results when the three "quadratic residue powers" $\rho^1, \rho^4, \rho^2$ of the primitive 7th root of unity, $\rho = e^{2\pi i/7}$, are connected by line segments in the complex plane. This suggests generalizations of the heptagonal triangle to the convex polygons spanning the $r$th-power-residues among the $n$th roots of unity, and relates the subject intimately to Gaussian Sums.

References

As mentioned in the text, expository material on the heptagonal triangle has until now been published only in French. References pertaining to problems that have appeared in American mathematical journals have been given in the body of this paper. The sources in French are found in Mathe-isis, 1913–204; 1938–169; 1950–344; 1955–78 and 329; 1956–106 and 149.