The Super Edge-Gracefulness of Two Infinite Families of Trees

Sin-Min Lee, Hugo Sun, Wandi Wei, Wen, and Paul Yiu

Abstract. For a positive integer $q$, let $L(q)$ be the set of $k$ integers, smallest in absolute value, and symmetric about 0. A connected, simple $(p, q)$-graph $G = (V, E)$ is said to be super edge-graceful if there is a bijection $f : E \rightarrow L(q)$ inducing a bijection $f^* : V \rightarrow L(p)$ via $f^*(u) = \sum_{\{u, v\} \in E} f(u, v)$. Let $T(n; (a_1, a_2, \ldots, a_n))$ be the tree obtained by amalgamating the path $P_n$ at each vertex $u$, $t = 1, 2, \ldots, n$, with a path of length $a_t$. We validate a conjecture of Lee and Wei that the trees $T(2m + 3; (02^{2^m}+0))$ and $T(2m + 2; (02^{2^m}1))$ are super edge-graceful by giving in each case a large lower bound, exponential in $m$, of the number of super edge-graceful labelings of the tree.

Key words and phrases: edge-graceful, super edge-graceful, trees, graph labelings, amalgamation, $S$-triplet partition.

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1. Introduction

In this article we study the super edge-gracefulness of some families of graphs and give a large lower bound of the number of super edge-graceful labelings for each of them. The notion of a super edge-graceful graph was introduced by J. Mitchem and A. Simoson [8]. Let $Z$ be the set of integers. For $a, b \in Z$, let

$$Z[a, b] := \{z \in Z : a \leq z \leq b\}.$$ 

In an obvious way we define its counterparts when $Z$ is replaced by $\mathbb{N}$ (the set of natural numbers), $\mathbb{N}_0$ (nonnegative integers), $\mathbb{O}$ (odd integers), and $\mathbb{E}$ (even integers). Note that some values of $a, b$ may make some of these sets the empty set.

For any $k \in \mathbb{N}$, let

$$L(k) = \begin{cases} \mathbb{Z} \left[-\frac{k-1}{2}, \frac{k-1}{2}\right], & \text{when } k \text{ is odd,} \\ \mathbb{Z} \left[-\frac{k}{2}, \frac{k}{2}\right] \setminus \{0\}, & \text{when } k \text{ is even.} \end{cases}$$

Thus, $L(k)$ consists of the $k$ integers, smallest in absolute value, symmetric with respect to 0.

By a $(p, q)$-graph we mean a connected, simple graph with $p$ vertices and $q$ edges. Such a graph $G = (V, E)$ is said to be super edge-graceful, SEG for short, if there exists a bijection $f : E \rightarrow L(q)$ such that the induced mapping $f^* : V \rightarrow \mathbb{Z}$
defined by
\[ f^*(u) = \sum_{\{u,v\} \in E} f(u,v) \]  
(1)
is a bijection onto \( L(p) \). Such a bijection \( f \) is called an SEG labeling of \( G \), \( f(e) \ (e \in E) \) the label of \( e \) under \( f \), and \( f^*(u) \ (u \in V) \) the label of \( u \) under \( f \).

**Example 1.** If \( n = 2k \), the path \( P_n \) (with \( n \) edges and \( n+1 \) vertices) is super edge-graceful. Here is an SEG labeling. We label the first half the vertices beginning with \( k \), and decreasing steadily by 1 to the “central” vertex 0. The edge labels, beginning with \( k \) and \(-1\), decrease steadily 1 for every two consecutive edges, to immediately preceding the vertex 0. The labels in the remaining half is the reflection of this in 0, with reversal of sign in every vertex and edge label. See Figure 1. Note that by joining the two end vertices (with labels \( k \) and \(-k\)) with an edge labeled 0, we obtain a super edge-graceful labeling of the cycle \( C_{2k+1} \).

![Figure 1. SEG labeling of \( P_{2k} \)](image)

On the other hand, it is easy to check that \( P_3 \) and \( P_5 \) (with 4 and 6 vertices respectively) are not super edge-graceful. More generally, trees of order 4 and 6 are not super edge-graceful; see [2]. Figure 2, however, shows that the \( P_7 \) and \( P_9 \) are.

\( P_7 \):

![Figure 2. SEG labelings of \( P_7 \) and \( P_9 \)](image)
Example 2. It is shown in [5] that the five trees of order 8 in Figure 3 are not super edge-graceful.

The concept of an SEG graph is closely related to the one of an edge-graceful graph introduced by S. P. Lo [7]. A \((p, q)\)-graph \(G = (V, E)\) is said to be edge-graceful if there exists a bijection \(f' : E \rightarrow \mathbb{N}[1, q]\) such that its induced mapping \((f')^* : V \rightarrow \mathbb{N}\) defined by

\[
(f')^*(u) = \sum_{\{u,v\} \in E} f'(u, v) \mod p
\]

is a bijection onto \(\mathbb{N}[0, p - 1]\), where \(a \mod p\) stands for the least nonnegative remainder of \(a \in \mathbb{Z}\) divided by \(p\). Mitchem and Simoson [8] have shown that a super edge-graceful \((p, q)\)-graph is edge-graceful if

\[
q \equiv \begin{cases} 
-1 \pmod{p} & \text{when } q \text{ is even}, \\
0 \pmod{p} & \text{when } q \text{ is odd}.
\end{cases}
\]

Therefore, any super edge-graceful tree of odd order is edge-graceful. It is also known that any tree of odd order with diameter at most four is edge-graceful. It is not true, however, that every SEG graph is edge-graceful; see [1, p.124].
2. The tree $T(n; (a_1, a_2, \ldots, a_n))$

Given $n \geq 3$ and a list of nonnegative integers $(a_1, a_2, \ldots, a_n)$, we consider, following [3], the tree $T(n; (a_1, a_2, \ldots, a_n))$ obtained by amalgamating $P_{n-1}$ (with $n$ vertices) at each vertex $u_t$, $t = 1, \ldots, n$, with a path of length $a_t$.

If $a_2 = \cdots = a_{n-1} = a$, we simply denote the tree by $T(n; (a_1a^{n-2}a_n))$. For example, Figure 5 shows the tree $T(n + 2; 02^n0)$. 

Figure 4. The tree $T(n; (a_1, a_2, \ldots, a_n))$

Figure 5. The tree $T(n + 2; 02^n0)$
This shorthand notation applies to repetitions of blocks as well. For example, in [6, Theorems 3.1 and 4.1], it is shown that the trees $T(4k + 3; (0(01)^{2k}00))$ and $T(2k + 3; (00)^{2k}00))$ are super edge-graceful. The following conjecture is also proposed in [6].

**Conjecture.** The graphs in the two infinite families $T(2m + 3; (02^{2m+1}0))$ and $T(2m + 2; (02^{2m}1))$ are super edge-graceful.

In this article, we validate this conjecture by establishing very large lower bounds for the numbers of super edge-graceful labelings of these trees. Denote by $N(G)$ the number of SEG labelings of a graph $G$. This is positive if and only if $G$ is super edge-graceful.

**Theorem 1.** $N(T(2m + 3; (02^{2m+1}0))) \geq 2^{2m+3} \cdot 3^m \cdot m!$.

**Theorem 2.** $N(T(2m + 2; (02^{2m}1))) \geq 2^{2m+1} \cdot 3^m \cdot m!$.

Before giving the proofs of these theorems, we present some preliminary results.

### 3. S-triplet partitions

We review the notion of $S$-triplet partition from [6], with slightly different formulation, and restate some results with clearer proofs. A 3-element set of positive integers (unordered) is called an $S$-triplet if one element is the sum of the remaining two. Let $A \subset \mathbb{N}$. A partition of $A$ is called an $S$-triplet partition if every part is an $S$-triplet. We say that such a set is $S$-triplet partitionable. The following examples play crucial roles in the proofs of our main Theorems 1 and 2.

**Proposition 3.** For $t \in \mathbb{N}$, the set $A := \mathbb{N}[1, 6t + 1] \setminus \{5t + 1\}$ has an $S$-triplet-partition

$$A = \bigcup_{i=1}^{t} \left( \{2i - 1, 3t - i + 1, 3t + i\} \cup \{2i, 5t - i + 1, 5t + i + 1\} \right).$$

**Proof.** Each 3-element set in (2) is an $S$-triplet, the third element being the sum of the remaining two. Now we prove that (2) is a partition. For $i = 1, 2, \ldots, t$, let

$$(a_{i1}, a_{i2}, a_{i3}) = (2i - 1, 3t - i + 1, 3t + i),$$

$$(a'_{i1}, a'_{i2}, a'_{i3}) = (2i, 5t - i + 1, 5t + i + 1).$$

Note that

$$\{a_{i1} : 1 \leq i \leq t\} = \mathbb{N}[1, 2t - 1] = \mathbb{N}[1, 2t],$$

$$\{a_{i2} : 1 \leq i \leq t\} = \mathbb{N}[2t + 1, 3t],$$

$$\{a_{i3} : 1 \leq i \leq t\} = \mathbb{N}[3t + 1, 4t],$$

$$\{a'_{i1} : 1 \leq i \leq t\} = \mathbb{N}[2, 2t] = \mathbb{N}[1, 2t],$$

$$\{a'_{i2} : 1 \leq i \leq t\} = \mathbb{N}[4t + 1, 5t],$$

$$\{a'_{i3} : 1 \leq i \leq t\} = \mathbb{N}[5t + 2, 6t + 1].$$
These sets are pairwise disjoint and
\[
\begin{align*}
\{a_{i_1} : 1 \leq i \leq t\} \cup \{a'_{i_1} : 1 \leq i \leq t\} &= \mathbb{O}[1, 2t] \cup \mathbb{E}[1, 2t] \\
&= \mathbb{N}[1, 2t], \\
\{a_{i_2} : 1 \leq i \leq t\} \cup \{a'_{i_2} : 1 \leq i \leq t\} &= \mathbb{N}[2t + 1, 3t] \cup \mathbb{N}[3t + 1, 4t] \\
&= \mathbb{N}[2t + 1, 4t], \\
\{a'_{i_2} : 1 \leq i \leq t\} \cup \{a'_{i_3} : 1 \leq i \leq t\} &= \mathbb{N}[4t + 1, 5t] \cup \mathbb{N}[5t + 2, 6t + 1] \\
&= \mathbb{N}[5t + 2, 6t + 1].
\end{align*}
\]

It is clear that their union is \(\mathbb{N}[1, 6t + 1] \setminus \{5t + 1\}\). \(\square\)

**Proposition 4.** For \(t \in \mathbb{N}\), the set \(B := \mathbb{N}[1, 6t + 4] \setminus \{5t + 4\}\) has an \(S\)-triplet-partition
\[
B = \left( \bigcup_{i=1}^{t} (\{2i - 1, 3t - i + 3, 3t + i + 2\} \cup \{2i, 5t - i + 4, 5t + i + 4\}) \right) \\
\cup \{2t + 1, 2t + 2, 4t + 3\}.
\] (3)

**Proof.** Each 3-element set in (3) is an \(S\)-triplet, the third element being the sum of the remaining two. Now we prove that (3) is a partition. For \(i = 1, 2, \ldots, t\), let
\[
\begin{align*}
(b_{i_1}, b_{i_2}, b_{i_3}) &= (2i - 1, 3t - i + 3, 3t + i + 2), \\
(b'_{i_1}, b'_{i_2}, b'_{i_3}) &= (2i, 5t - i + 4, 5t + i + 4).
\end{align*}
\]
Also let
\[
(b_{t+1,1}, b_{t+1,2}, b_{t+1,3}) = (2t + 1, 2t + 2, 4t + 3).
\]

Then we have
\[
\begin{align*}
\{b_{i_1} : 1 \leq i \leq t + 1\} &= \mathbb{O}[1, 2t + 1], \\
\{b_{i_2} : 1 \leq i \leq t + 1\} &= \mathbb{N}[2t + 2, 3t + 2], \\
\{b_{i_3} : 1 \leq i \leq t + 1\} &= \mathbb{N}[3t + 3, 4t + 3], \\
\{b'_{i_1} : 1 \leq i \leq t\} &= \mathbb{E}[2, 2t] = \mathbb{E}[1, 2t + 1], \\
\{b'_{i_2} : 1 \leq i \leq t\} &= \mathbb{N}[4t + 4, 5t + 3], \\
\{b'_{i_3} : 1 \leq i \leq t\} &= \mathbb{N}[5t + 5, 6t + 4].
\end{align*}
\]
These sets are pairwise disjoint and
\[
\begin{align*}
\{b_{i1} : 1 \leq i \leq t + 1\} &\cup \{b'_{i1} : 1 \leq i \leq t\} \\
= \mathbb{N}[1, 2t + 1] &\cup E[1, 2t + 1] \\
= \mathbb{N}[1, 2t + 1],
\end{align*}
\]
\[
\begin{align*}
\{b_{i2} : 1 \leq i \leq t + 1\} &\cup \{b_{i3} : 1 \leq i \leq t + 1\} \\
= \mathbb{N}[2t + 2, 3t + 2] &\cup \mathbb{N}[3t + 3, 4t + 3] \\
= \mathbb{N}[2t + 2, 4t + 3],
\end{align*}
\]
\[
\begin{align*}
\{b'_{i2} : 1 \leq i \leq t\} &\cup \{b'_{i3} : 1 \leq i \leq t\} \\
= \mathbb{N}[4t + 4, 5t + 3] &\cup \mathbb{N}[5t + 5, 6t + 4] \\
= \mathbb{N}[4t + 4, 6t + 4] \setminus \{5t + 4\}.
\end{align*}
\]

The union of these sets is clearly \(\mathbb{N}[1, 6t + 1] \setminus \{5t + 4\}\). \(\Box\)

Combining these two theorems, we immediately have

**Theorem 5.** For \(m \in \mathbb{N}\), let

\[
c = \begin{cases} 
5t + 1, & \text{if } m = 2t, \\
5t + 4, & \text{if } m = 2t + 1.
\end{cases}
\] (4)

The set \(C := \mathbb{N}[1, 3m + 1] \setminus \{c\}\) is \(S\)-triplet-partitionable.

### 4. Enumeration of labelings of two basic graphs

4.1. The graph \(T = T(3; (0, 2, 2))\). The graph \(T = T(3; (0, 2, 2))\) is super edge-graceful; see Figure 6 with an SEG labeling. We enumerate all such labelings satisfying (1) without restricting the edge labels to \(L(6) = \{\pm 1, \pm 2, \pm 3\}\). Let \(a, b, c \in \mathbb{N}\) satisfy \(a + b = c\). We label the edges of \(T\) by \(\{\pm a, \pm b, \pm c\}\) and the vertices by \(\{0, \pm a, \pm b, \pm c\}\), with \(f^*(v_3) = 0\). If \(f\) and \(f^*\) satisfies (1), we call the resulting labeling of \(T\) a \(G\)-labeling from the \(S\)-triplet \(\{a, b, c\}\).

![Figure 6. The graph T and an SEG labeling](image)

**Proposition 6.** Let \(a, b, c \in \mathbb{N}\) satisfy \(a + b = c\). The graph \(T\) has at least 12 \(G\)-labelings from the \(S\)-triplet \(\{a, b, c\}\).
**Proof.** We begin with three distinct $G$-labelings $f_1$, $f_2$, $f_3$ of $T$.

![Image of three G-labelings](image)

Figure 7. Three $G$-labelings of $T$

Let us define two unary operations $P$ and $N$ on the $G$-labelings.

(i) $Pf$ is the $G$-labeling induced by the permutation simultaneously interchanging the values $a$, $b$ and $-a$, $-b$. Figure 8 shows the effect of $P$ on the labelings $f_1$, $f_2$, $f_3$.

(ii) $Nf$ is the $G$-labeling induced by the reversal of signs of the values under $f$. Figure 9 shows the effect of $N$ on the labelings $f_1$, $f_2$, $f_3$.

![Image of three G-labelings from P](image)

Figure 8. Three $G$-labelings of $T$ from $P$

Figure 10 shows the labelings $NPf_i$, $i = 1, 2, 3$. It is clear that the twelve labelings given in Figures 7, 8, 9, 10 are distinct. This completes the proof of the proposition. \[\square\]

Though Proposition 6 is sufficient for the proof of Theorems 1 and 2, it can indeed be strengthened.

(1) If $a < b < c$ are natural numbers satisfying $a + b = c$ and $b \neq 2a$, there are no other $G$-labeling of $T$ by the $S$-triplet $\{a, b, c\}$ apart from those given in the proof of Proposition 6.

**Proof.** Suppose to the contrary that there is a different $G$-labeling $f$, with induced bijection $f^*$. Since

$$f^*(v_3) = 0$$

(5)
we have
\[ f(\{v_3, v_1, v_3\}) = -f(\{v_2, v_3\}). \] (6)
Since \( Pf, Nf \) and \( NP \) are also \( G \)-labelings, we may assume without loss of generality that (6) is realized by
\[ f(\{v_3, v_1, v_3\}) = a, \ f(\{v_2, v_3\}) = -a \] (7)
or
\[ f(\{v_3, v_1, v_3\}) = c, \ f(\{v_2, v_3\}) = -c. \] (8)
In case of (7), ??
I do not have the diagram to complete the reasoning. □

(2) If \( a < b < c \) are natural numbers satisfying \( a + b = c \) and \( b = 2a \), there are at least 4 more \( G \)-labelings of \( T \) by the \( S \)-triplet \( \{a, b, c\} \) apart from those given in the proof of Proposition 6.

Proof. Figures 11 and 12 show one more \( G \)-labeling \( f_4 \) and and the effect of the unary operations on it. These give four more \( G \)-labelings distinct from those in Figures 7, 8, 9, and 10. □
Figure 11. A $G$-labeling $f_4$ of $G$ in the case $b = 2a$

Figure 12. Three more $G$-labelings of $T$ from $f_4$

4.2. The graph $T' = T(3; (0, 2, 0))$. Now we consider another super edge-graceful graph $T' = T(3; (0, 2, 0))$:

Again we consider more general labelings of edges by $\{\pm a, \pm b\}$, and vertices by $\{0, \pm a, \pm b\}$. If a bijection $f : E \to \{\pm a, \pm b\}$ induces a bijection $f^* : V \to \{0, \pm a, \pm b\}$ satisfying (1), we call it a $K$-labeling of $T'$ from the pair $\{a, b\}$. 
Proposition 7. Let $a, b \in \mathbb{Z}$ with $a \neq \pm b$. There are at least 8 distinct $K$-labelings of $T'$ from the pair $\{a, b\}$.

Proof. We begin with a $K$-labeling given in Figure 14 below.

\[
\begin{array}{c}
\text{\textbf{Figure 14. A $K$-labeling of $T'$}} \\
\end{array}
\]

Let us define three unary operations $P$, $A$ and $B$ on $K$-labelings.

(i) $Pf$ is the $K$-labeling induced by the permutation simultaneously interchanging the values $a$, $b$ and $-a$, $-b$.

(ii) $Af$ is the $K$-labeling induced by interchanging the values $a$ and $-a$.

(iii) $Bf$ is the $K$-labeling induced by interchanging the values $b$ and $-b$.

Applying these operations to the labeling $f$, we have the three in Figure 15.

\[
\begin{array}{c}
\text{\textbf{Figure 15. Three unary operations on a $K$-labeling}} \\
\end{array}
\]

Figure 16 displays four more labelings $APf$, $BPf$, $ABf$ and $BAPf$. All these are $K$-labelings distinct from $f$. This proves the proposition. \qed
5. Proof of Theorem 1

The tree $T(2m + 3, (02^{2m+1}0))$ has $p = 6m + 5$ vertices and $q = 6m + 4$ edges. It can be regarded as an amalgamation of $m$ copies of $T$ and one copy of $T'$.
Note that \( L(q) = \mathbb{Z} \setminus (3m + 2, 3m + 2] \setminus \{0\} \). By Theorem 5, the set \( C = \mathbb{N}[1, 3m + 1] \setminus \{c\} \) is \( S \)-triplet-partitionable, where \( c \) is given by (4).

Let \( \{S_j : 1 \leq j \leq m\} \) be an \( S \)-triplet partition of \( C \). We give each copy of \( T_i \) a \( G \)-labeling from the \( S \)-triplet \( S_j \). There are \( m! \) possible distributions. For each of these, there are, by Proposition 6, at least 12 \( G \)-labelings. For the subgraph \( T' \), there are, by Proposition 7, at least 8 \( K \)-labelings from \( \{c, 3m + 2\} \). Therefore, there are at least \( 12^m \cdot m! \cdot 8 = 2^{2m+3} \cdot 3^m \cdot m! \) labelings of the tree. Since for each \( K \)-labeling of \( T \), \( f^*(v_3) = 0 \), superpositions of the labelings of the components give an SEG labeling of the tree. This completes the proof of Theorem 1.

Figure 18 gives a super edge-graceful labeling of \( T(7; (02^5 0)) \).

6. Proof of Theorem 2

The tree \( T(2m + 2, (02^{2m} 1)) \) has \( p = 6m + 3 \) vertices and \( q = 6m + 2 \) edges. It can be regarded as an amalgamation of \( m \) copies of \( T \) and one copy of \( P_3 \). See Figure 19.
In this case, \( L(q) = \mathbb{Z}[-(3m+1), 3m+1] \setminus \{0\} \). Consider as above an \( S \)-triplet partition \( \{ S_j : 1 \leq j \leq m \} \) of \( C = \mathbb{N}[1, 3m + 1] \setminus \{c\} \). Again, there are \( m! \) possible distributions of these triplets to the \( m \) copies of \( T \), and for each of these, there are at least 12 \( G \)-labelings. On the other hand, there are obviously 2 ways of labeling the edges of the \( P_3 \) component with \( \{c, -c\} \). With the induced vertex labelings, these are shown in Figure 20.

![Figure 20. Labelings of \( P_3 \)](attachment:image.png)

Therefore, there are at least \( 12^m \cdot m! \cdot 2 = 2^{2m+1} \cdot 3^m \cdot m! \) labelings of the tree. Since for each \( K \)-labeling of \( T \), \( f^*(v_3) = 0 \), superpositions of the labelings of the components give an SEG labeling of the tree. This completes the proof of Theorem 2.

Figure 21 gives a super edge-graceful labeling of \( T(6; (02^11)) \).

![Figure 21. A super edge-graceful labeling of \( T(6; (02^11)) \)](attachment:image.png)

References

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Sin Min Lee: Department of Computer Science, San Jose State University, San Jose, California
95192 USA

Hugo Sun:

Wandi Wei: Center for Cryptology and Information Security, Florida Atlantic University, Boca
Raton, Florida 33431 USA

Wen:

Paul Yiu: Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida
33431 USA