Heron Sequences and Their Modifications

Paul Yiu, [1] K. R. S. Sastry [2] and Shanzhen Gao [3]

1. Heron sequences

Heron triangles are triangles with integer sides and integer areas. They have been studied since ancient times. For some recent studies, see for examples [1, 2, 3, 4, 6, 7]. It is clear that the angles of a Heron triangle all have rational sines and cosines. Such are called Heron angles, see for example [4]. Conversely, if the angles of a triangle are all Heron angles, then an appropriate magnification yields a Heron triangle. In this paper we study natural number sequences associated with Heron triangles. One most natural and interesting question is whether there is an infinite increasing sequence in which every three consecutive terms are the sidelengths of a Heron triangle. While we do not know the answer to this question, we can prove that such a sequence can be arbitrarily long. The proof makes use of the Fibonacci numbers F_n defined recursively by

$$F_{n+1} = F_n + F_{n-1}, \quad F_1 = 1, \ F_2 = 1.$$

Theorem 1. *Given an integer $n \geq 3$, there is a sequence*

$$a_1, \ a_2, \ \ldots, a_n$$

every three consecutive terms of which are the sides of a Heron triangle.

[1]Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida 33431-0991, USA; email: yiu@fau.edu

[2]Jeevan Sandhya, DoddaKalsandra Post, Raghuvana Halli, Bangalore, 560 062, India.

[3]Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida 33431-0991, USA; email: sgao2@fau.edu

Proof. Let θ be a Heron angle such that $F_{n+1}\theta < \frac{\pi}{2}$. Consider the points

$$P_k = (\cos 2F_{k+2}\theta,\ \sin 2F_{k+2}\theta),$$

for $k = 1, \ldots, n+1$. These are rational points on the unit circle. For $k = 1, \ldots, n$, the length of the chord P_kP_{k+1}

$$a_k = 2\sin\frac{2F_{k+3} - 2F_{k+2}}{2}\theta = 2\sin F_{k+1}\theta \in \mathbb{Q}.$$

Since $F_{n+1}\theta < \frac{\pi}{2}$, the sequence of rational numbers $a_1,\ a_2,\ \ldots, a_n$ is strictly increasing. We claim that every three consecutive terms are the sides of a triangle with rational area. To see this, note that for $k = 3, \ldots, n$, two of the sides of triangle $P_{k-2}P_{k-1}P_k$ have lengths a_{k-2} and a_{k-1}. The length of the third side $P_{k-2}P_k$ is

$$2\sin\frac{2F_{k+2} - 2F_k}{2}\theta = 2\sin F_{k+1}\theta = a_k.$$

Since the vertices of the triangle $P_{k-2}P_{k-1}P_k$ are on the unit circle, its area is $\frac{a_{k-2}a_{k-1}a_k}{4}$, a rational number. By clearing denominators, we obtain an n-term sequence of integers every three consecutive terms of which form a Heron triangle. □

The numbers realizing the Heron triangles in the above construction increase very rapidly in n. By a routine computer search beginning with two positive integers < 1000, we have found that the longest Heron sequence contains 9 terms. There are three such sequences. One of these is

sides	60	275	325	500	525	697	746	1345	1797
area			4950	41250	78750	130872	175644	175644	452844

This sequence has been recorded in the *Encyclopedia of Integer Sequences* as sequence A134587.

The other two 9-term sequences are obtained by multiplying the sides in this sequence by 2 and 3 respectively. Again, beginning with two integers < 1000, there are two 8-term modified Hereon sequences:

sides	445	485	850	1095	1435	2516	2691	3505
area			80100	197100	464100	165648	1782270	3369960

sides	825	975	1500	1575	2091	2238	4035	5391
area			371250	708750	1177848	1580796	1580796	4075596

2. MODIFIED HERON SEQUENCES

A common way to construct Heron triangles is to choose three integers u, v, w such that

$$uvw(u+v+w) = \triangle^2 \tag{1}$$

for an integer $\triangle$. The triangle with sides

$$a = u+v, \qquad b = u+w, \qquad c = v+w$$

is a Heron triangle with area $\triangle$. By a modified Heron sequence we mean an *increasing* sequence (u_n) of positive integers such that every three consecutive terms u_{n-2}, u_{n-1} and u_n determine a Heron triangle with sides

$$a_n = u_{n-2} + u_{n-1}, \quad b_n = u_{n-2} + u_n, \qquad c_n = u_{n-1} + u_n,$$

and integer area $\triangle_n$. While it is not *a priori* obvious that a Heron sequence may be infinite, we show that, apart from a few exceptions, an infinite modified Heron sequence always results if one begins with two distinct positive integers.

Theorem 2. *Let $u < v$ be positive integers such that*

$$(u,v) \neq (1,4),\ (1,9),\ (2,8),\ (2,18),\ (4,16). \tag{2}$$

There is an infinite modified Heron sequence (u_n) satisfying
(i) $u_{-1} = u$, $u_0 = v$;
(ii) *u_n is the smallest positive integer $> u_{n-1}$ such that the triangle determined by u_{n-2}, u_{n-1}, and u_n is a Heron triangle.*

Proof. Given two positive integers $u < v$, to find an integer w which together with u and v gives a Heron triangle we solve (1) by rewriting it in the form

$$x^2 - uv \cdot y^2 = -uv(u+v)^2 \tag{3}$$

by putting $x = 2\triangle$ and $y = 2w + u + v$.

The case uv not equal to a square. It is readily seen that $(x, y) = (0, u + v)$ is a solution of (3). If uv is not the square of an integer, then there is an infinite sequence of integer solutions generated by the fundamental solution of the Fermat-Pell equation

$$x^2 - uv \cdot y^2 = 1. \tag{4}$$

Namely, if (a, b) is the smallest positive solution of (4), then there is a sequence $(x_n,\ y_n)$ of solutions of (3) given by

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a & uvb \\ b & a \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \qquad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ u+v \end{pmatrix}. \tag{5}$$

Note that $a \geq 3$ except when $uv = 3$. In particular, $y_1 = a(u + v) \geq 3(u + v)$. We can determine an integer w from $y_1 = 2w + u + v$ which satisfies $w \geq u + v > v$. For $(u, v) = (1, 3)$, the solution y_2 would give $w > v$.

The case uv equal to a square. If uv is a square, equation (3) reduces to

$$(2w + u + v)^2 = (u + v)^2 + z^2 \tag{6}$$

for some integer z. This requires $u + v$ to be a shorter side of a Pythagorean triangle whose hypotenuse is $2w + u + v$ for some integer $w > v$. Now, it is well known (and easy to verify) that every integer ≥ 3 is a shorter side of a Pythagorean triangle. However, to ensure a hypotenuse of sufficient length, we examine the details. Note that we require the hypotenuse to be $2w + u + v$, which has the same parity as $u + v$.

(i) If $u+v=4k$ for some integer k, consider the Pythagorean triangle

$$(4k,\ 2(k^2-1),\ 2(k^2+1)).$$

If we set the hypotenuse to be $2w+4k$, then $w=(k-1)^2$. If $k\geq 6$, $w=(k-1)^2>4k>v$. The only pairs (u,v) which do not satisfy this condition are $(u,v)=(2,18)$ and $(4,16)$.

(ii) If $u+v=4k+2$ for some integer k, consider the Pythagorean triangle

$$(2(2k+1),\ (2k+1)^2-1,\ (2k+1)^2+1).$$

If we set $2w+4k+2=(2k+1)^2+1$, then $w=2k^2$. If $k\geq 3$, then $w=2k^2>4k+2>v$. The only pairs (u,v) which do not satisfy this condition are $(1,9)$ and $(2,8)$.

(iii) If $u+v=4k+1$ for some integer k, consider the Pythagorean triangle

$$(4k+1,\ 4k(2k+1),\ (2k+1)^2+(2k)^2).$$

If we set $2w+4k+1=(2k+1)^2+(2k)^2$, then $w=4k^2$. If $k\geq 2$, $w=4k^2>4k+1>v$. The only pair (u,v) which does not satisfy this condition is $(1,4)$.

(iv) If $u+v=4k+3$ for some integer k, consider the Pythagorean triangle

$$(4k+3,\ 4(k+1)(2k+1),\ (2k+1)^2+(2k+2)^2).$$

If we set $2w+4k+3=(2k+1)^2+(2k+2)^2$, then $w=(2k+1)^2$, which always exceeds $4k+3$ for $k\geq 1$.

It follows that if uv is a square, then apart from the pairs in the list (2), there exists an integer $w>v$ such that u, v, w determine a Heron triangle.

For (u,v) given in the list (2), it is routine to check that Pythagorean triangle in (6) is an integer multiple of $(5,12,13)$. In each case, the corresponding value of w is smaller than v.

Therefore, beginning with $(u_{-1},\ u_0)$ which is not one of the pairs in the list (2) we can determine inductively for $n\geq 1$, u_n as the smallest integer $>\ u_{n-1}$ such that u_{n-2}, u_{n-1}, u_n

determine a Heron triangle. The sequence so obtained is infinite. □

Remark. The above constructions may or not not give the smallest possible w in question. However, the existence of the smallest $w > v$ is guaranteed.

We conclude with some examples of modified Heron sequences each determined by two initial entries. The first three have been recorded in the *Encyclopedia of Integer Sequences*

EIS number	Modified Heron sequence
$A134588$	$1, 2, 3, 10, 27, 98, 120, 327, \ldots$
$A134589$	$1, 3, 12, 49, 108, 243, 624, \ldots$
$A134590$	$2, 5, 63, 112, 140, 315, 364, \ldots$
	$6, 7, 8, 27, 70, 750, 972, \ldots$

The Heron triangles determined by the last sequence are

$$(13, 14, 15; 84),\ (15, 34, 35; 252),\ (35, 78, 97; 1260),$$
$$(97, 777, 820; 34650),\ (820, 1042, 1722; 302400), \ldots.$$

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