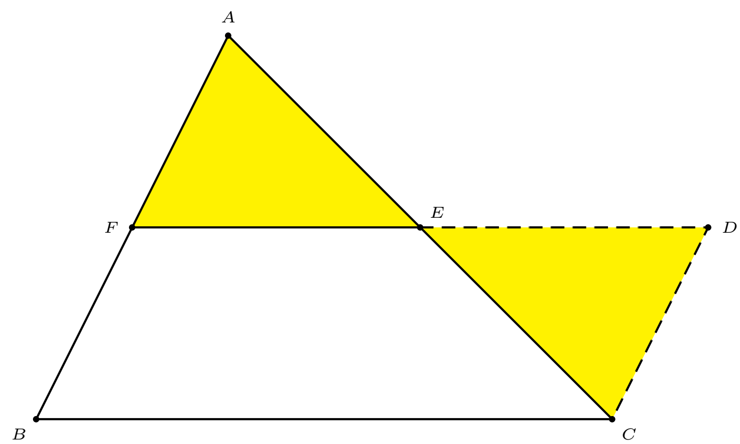
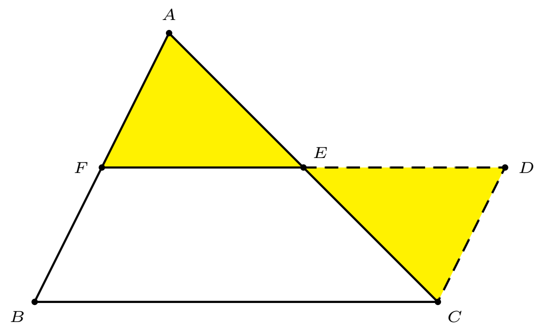
The midpoint theorem and its converse

Theorem (Midpoint theorem)**.** Given triangle ABC, let E and F be the midpoints of AC and AB respectively. The segment FE is parallel to BC and its length is one half of the length of BC.



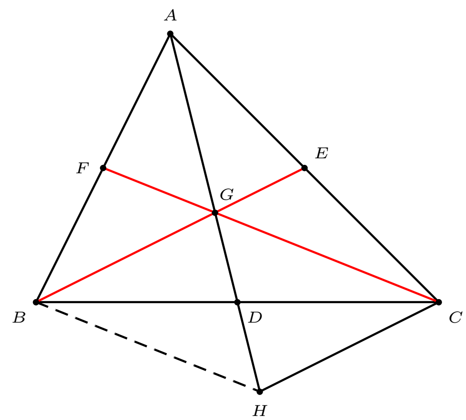
Proof. Extend FE to D such that $FE = ED$.
Note that $\triangle CDE \equiv \triangle AFE$ by the ____________ test.
It follows that
(i) $CD = AF = BF$,
(ii) $\angle CDE = \angle AFE$, and $CD//BA$.
Therefore, $CDFB$ is a parallelogram, and $FE//BC$.
Also, $BC = FD = 2FE$. $\square$

Theorem (Converse of midpoint theorem)**.** Let F be the midpoint of the side AB of triangle ABC. The parallel through F to BC intersects AC at its midpoint.



Proof. Construct the parallel through C to AB,
and extend FE to intersect this parallel at D.
Then, $CDFB$ is a parallelogram,
and $CD = BF = FA$.
It follows that $\triangle AEF \equiv CED$ by the ____________ test.
This means that $AE = CE$, and E is the midpoint of AC. □

Theorem. The three medians of a triangle are concurrent.



Proof. Let E and F be the midpoints of AC and AB respectively,
and G the intersection of the **medians** BE and CF.
Construct the parallel through C to BE,
and extend AG to intersect BC at D, and this parallel at H.
By the converse of the midpoint theorem,
G is the midpoint of AH, and $HC = 2 \cdot GE$.
Join BH.
By the midpoint theorem, $BH//CF$.
It follows that $BHCG$ is a parallelogram.
Therefore, D is the midpoint of (the diagonal) BC,
and AD is also a median of triangle ABC.
Therefore, the three medians AD, BE, CF concur at G. □

The point of concurrency of the three medians of a triangle is called the **centroid** of the triangle.
Note that

$$\begin{aligned} AG =& GH = 2GD, \\ BG =& HC = 2GE, \\ CG =& HB = 2GF. \end{aligned}$$