Chapter 25

Archimedes’ quadrature of the parabola

**Proposition** (Apollonius I.11). *If a cone is cut by a plane through its axis, and also cut by another plane cutting the base of the cone in a straight line perpendicular to the base of the axial triangle, and if, further, the diameter of the section is parallel to one side of the axial triangle, and if any straight line is drawn from the section of the cone to its diameter such that this straight line is parallel to the common section of the cutting plane and of the cone’s base, then this straight line to the diameter will equal in square the rectangle contained by
(a) the straight line from the section’s vertex to where the straight line to the diameter cuts it off and
(b) another straight line which has the same ratio to the straight line between the angle of the cone and the vertex of the section as the square on the base of the axial triangle has to the rectangle contained by the remaining two sides of the triangle. And let such a section be called a parabola.*
25.1 Basic properties of the parabola from Archimedes

In segments bounded by a straight line and any curve I call the straight line the base, and the height the greatest perpendicular drawn from the curve to the base of the segment, and the vertex the point from which the greatest perpendicular is drawn.

**Proposition (QP 1).** If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as $PV$, and if $QQ'$ be a chord parallel to the tangent to the parabola at $P$ and meeting $PV$ in $V$, then

$$QV = VQ'.$$

Conversely, if $QV = VQ'$, the chord $QQ'$ will be parallel to the tangent at $P$.

![Diagram](attachment:parabola.png)

**Proposition (QP 2).** If in a parabola $QQ'$ be a chord parallel to the tangent at $P$, and if a straight line be drawn through $P$ which is either itself the axis or parallel to the axis, and which meets $QQ'$ in $V$ and the tangent at $Q$ to the parabola in $T$, then

$$PV = PT.$$
25.2 Quadrature of the Parabola

**Proposition (QP 3).** If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as PV, and if from two other points Q, Q' on the parabola straight lines be drawn parallel to the tangent at P and meeting PV in V, V' respectively, then

\[ PV : PV' = QV^2 : Q'V'^2. \]

And these propositions are proved in the elements of conics.

**Proposition (QP 5).** If Qq be the base of any segment of a parabola, P the vertex of the segment, and PV its diameter, and if the diameter of the parabola through any other point R meet Qq in O and the tangent at Q in E, then

\[ QO : Oq = ER : RO. \]

**Proposition (QP 17).** The area of any segment of a parabola is four-thirds of the triangle which has the same base as the segment and equal height.

**Proposition (QP 18).** If Qq be the base of a segment of a parabola, and V the middle point of Qq, and if the diameter through V meet the curve in P, then P is the vertex of the segment.

**Proposition (QP 19).** If Qq be a chord of a parabola bisected in V by the diameter PV, and if RM be a diameter bisecting QV in M, and RW be the ordinate from R to PV, then PV = \( \frac{4}{3}RM \).

**Proposition (QP 20).** If Qq be the base, and P the vertex of a parabolic segment, then the triangle PQq is greater than half the segment PQq.

**Proposition (QP 21).** If Qq be the base, and P the vertex, of any parabolic segment, and if R be the vertex of the segment cut off by PQ, then \( \triangle PQq = 8\triangle PRQ \).
Proposition (QP 22). If there be a series of areas \( A, B, C, D, \ldots \) each of which is four times the next in order, and if the largest, \( A \), be equal to the triangle \( PQq \) inscribed in a parabolic segment \( PQq \) and having the same base with it and equal height, then

\[
A + B + C + D + \cdots < \text{area of segment } PQq.
\]

Proposition (QP 23). Given a series of areas \( A, B, C, D, \ldots, Z \), of which \( A \) is the greatest, and each is equal to four times the next in order, then

\[
A + B + C + D + \cdots + Z + \frac{1}{3}Z = \frac{4}{3}A.
\]

Proposition (QP 24). Every segment bounded by the parabola and a chord \( Qq \) is equal to four-thirds of the triangle which has the same base as the segment and equal height.
Archimedes’ quadrature of the parabola
Chapter 26

Quadrature of the parabola in Archimedes’ Method

Proposition (Method, Proposition 1). Let $ABC$ be a segment of a parabola bounded by the straight line $AC$ and the parabola $ABC$, and let $D$ be the middle point of $AC$. Draw the straight line $DBE$ parallel to the axis of the parabola and join $AB$, $BC$. Then shall the segment $ABC$ be $\frac{4}{3}$ of the triangle $ABC$.

From $A$ draw $AKF$ parallel to $DE$, and let the tangent to the parabola at $C$ meet $DBE$ in $E$ and $AKF$ in $F$. Produce $CB$ to meet $AF$ in $K$, and again produce $CK$ to $H$, making $KH$ equal to $CK$.

Consider $CH$ as the bar of a balance, $K$ being its middle point.

Let $MO$ be any straight line parallel to $ED$, and let it meet $CF$, $CK$, $AC$ in $M$, $N$, $O$ and the curve in $P$.

Now, since $CE$ is a tangent to the parabola and $CD$ the semi-ordinate,

$$EB = BD;$$
“for this is proved in the Elements [of Conics].”

Since $FA$, $MO$ are parallel to $ED$, it follows that

$$FK = KA, \quad MN = NO.$$ 

Now, by the property of the parabola, “proved in a lemma,”

$$MO : OP = CA : AO = CK : KN = HK : KN.$$ 

Take a straight line $TG$ equal to $OP$, and place it with its center of gravity at $H$, so that $TH = HG$; then, since $N$ is the center of gravity of the straight line $MO$, and

$$MO : TG = HK : KN,$$

it follows that $TG$ at $H$ and $MO$ at $N$ will be in equilibrium about $K$.

[On the Equilibrium of Planes, I. 6,7]

Similarly, for all other straight lines parallel to $DE$ and meeting the arc of the parabola,

(1) the portion intercepted between $FC$, $AC$ with its middle point on $KC$ and

(2) a length equal to the intercept between the curve and $AC$ placed with its center of gravity at $H$

will be in equilibrium about $K$.

Therefore $K$ is the center of gravity of the whole system consisting

(1) of all the straight lines as $MO$ intercepted between $FC$, $AC$ and placed as they actually are in the figure, and

(2) of all the straight lines placed at $H$ equal to the straight lines as $PO$ intercepted between the curve and $AC$.

And, since the triangle $CFA$ is made up of all the parallel lines like $MO$, and the segment $CBA$ is made up of all the straight lines like $PO$ within the curve, it follows that the triangle, placed where it is in the figure, is in equilibrium about $K$ with the segment $CBA$ placed with its center of gravity at $H$.

Divide $KC$ at $W$ so that $CK = 3KW$; then $W$ is the center of gravity of the traignle $ACF$, “for this is proved in the books on equilibrium”.

[On the Equilibrium of Planes, I. 15].

Therefore, $\triangle ACF : \text{segment } ABC = HK : KW = 3 : 1$.

Therefore, segment $ABC = \frac{1}{3} \triangle ACF$.

But, $\triangle ACF = 4\triangle BAC$.

Therefore, segment $ABC = \frac{4}{3} \triangle ABC$.

Now the fact here stated is not actually demonstrated by the argument used; but that argument has given a sort of indication that the conclusion is true. Seeing

\[1\text{Heath’s footnote: i.e. the works on conics by Aristaeus and Euclid.}\]
then that the theorem is not demonstrated, but at the same time suspecting that the conclusion is true, we shall have recourse to the geometrical demonstration which I myself discovered and have already published.
Quadrature of the parabola in Archimedes’ Method
“for this is proved in the Elements [of Conics].”¹

Since \( FA, MO \) are parallel to \( ED \), it follows that

\[
FK = KA, \quad MN = NO.
\]

Now, by the property of the parabola, “proved in a lemma,”

\[
MO : OP = CA : AO = CK : KN = HK : KN.
\]

Take a straight line \( TG \) equal to \( OP \), and place it with its center of gravity at \( H \), so that \( TH = HG \); then, since \( N \) is the center of gravity of the straight line \( MO \), and

\[
MO : TG = HK : KN,
\]

it follows that \( TG \) at \( H \) and \( MO \) at \( N \) will be in equilibrium about \( K \).

[On the Equilibrium of Planes, I. 6,7]

Similarly, for all other straight lines parallel to \( DE \) and meeting the arc of the parabola,

1. the portion intercepted between \( FC, AC \) with its middle point on \( KC \) and
2. a length equal to the intercept between the curve and \( AC \) placed with its center of gravity at \( H \)

will be in equilibrium about \( K \).

Therefore \( K \) is the center of gravity of the whole system consisting

1. of all the straight lines as \( MO \) intercepted between \( FC, AC \) and placed as they actually are in the figure, and
2. of all the straight lines placed at \( H \) equal to the straight lines as \( PO \) intercepted between the curve and \( AC \).

And, since the triangle \( CFA \) is made up of all the parallel lines like \( MO \), and the segment \( CBA \) is made up of all the straight lines like \( PO \) within the curve, it follows that the triangle, placed where it is in the figure, is in equilibrium about \( K \) with the segment \( CBA \) placed with its center of gravity at \( H \).

Divide \( KC \) at \( W \) so that \( CK = 3KW \);

then \( W \) is the center of gravity of the traignle \( ACF \), “for this is proved in the books on equilibrium”.

[On the Equilibrium of Planes, I. 15].

Therefore, \( \triangle ACF : \text{segment} \ ABC = HK : KW = 3 : 1 \).

Therefore, segment \( ABC = \frac{1}{3} \triangle ACF \).

But, \( \triangle ACF = 4\triangle BAC \).

Therefore, segment \( ABC = \frac{4}{3} \triangle ABC \).

Now the fact here stated is not actually demonstrated by the argument used; but that argument has given a sort of indication that the conclusion is true. Seeing

¹Heath’s footnote: i.e. the works on conics by Aristaeus and Euclid.
then that the theorem is not demonstrated, but at the same time suspecting that the conclusion is true, we shall have recourse to the geometrical demonstration which I myself discovered and have already published.
Chapter 27

Archimedes’ calculation of the volume of a sphere

**Proposition** (Method, Proposition 2). (1) Any sphere is (in respect of solid content) four times the cone with base equal to a great circle of the sphere and height equal to its radius.

(2) The cylinder with base equal to a great circle of the sphere and height equal to the diameter is $1\frac{1}{2}$ times the sphere.

(1) Let $ABCD$ be a great circle of a sphere, and $AC$, $BD$ diameters at right angles to one another.

Let a circle be drawn about $BD$ as diameter and in a plane perpendicular to $AC$, and on this circle as base let a cone be described with $A$ as vertex. Let the surface of this cone be produced and then cut by a plane through $C$ parallel to its base; the
section will be a circle on $EF$ as diameter. On this circle as base let a cylinder be
erected with height and axis $AC$, and produce $CA$ to $H$, making $AH$ equal to $CA$.

Let $CH$ be regarded as the bar of a balance, $A$ being its middle point.

Draw any straight line $MN$ in the plane of the circle $ABCD$ and parallel to
$BD$. Let $MN$ meet the circle in $O, P$, the diameter $AC$ in $S$, and the straight lines
$AE, AF$ in $Q, R$ respectively. Join $AO$.

Through $MN$ draw a plane at right angles to $AC$; this plane will cut the cylinder
in a circle with diameter $MN$, the sphere in a circle with diameter $OP$, and the cone
in a circle with diameter $QR$.

Now, since $MS = AC$, and $QS = AS$,

$$MS \cdot SQ = CA \cdot AS = AO^2 = OS^2 + SQ^2.$$  

And, since $HA = AC$,

$$HA : AS = CA : AS = MS : SQ = MS^2 : MS \cdot SQ = MS^2 : (OS^2 + SQ^2) = MN^2 : (OP^2 + QR^2) = (\text{circle, diameter } MN) : (\text{circle, diameter } OP + \text{circle, diameter } QR).$$

That is, $HA : AS = (\text{circle in cylinder}) : (\text{circle in sphere + circle in cone})$.

Therefore the circle in the cylinder, placed where it is, is in equilibrium, about
$A$, with the circle in the sphere together with the circle in the cone, if both the latter
circles are placed with their centers of gravity at $H$.

Similarly for the three corresponding sections made by a plane perpendicular to
$AC$ and passing through any other straight line in the parallelogram $LF$ parallel to
$EF$.

If we deal in the same way with all the sets of three circles in which planes
perpendicular to $AC$ cut the cylinder, the sphere and the cone, and which make up
those solids respectively, it follows that the cylinder, in the place where it is, will be
in equilibrium about $A$ with the sphere and the cone together, when both are placed
with their centers of gravity at $H$.

Therefore, since $K$ is the centre of gravity of the cylinder,

$$HA : AK = (\text{cylinder}) : (\text{sphere + cone } AEF).$$

But $HA = 2AK$; therefore cylinder $= 2$ (sphere + cone $AEF$). Now cylinder $= 3$
(cone $AEF$); $^1$ therefore cone $AEF = 2$ (sphere). But, since $EF = 2BD$, cone
$AEF = 8$ (cone $ABD$); therefore sphere $= 4$ (cone $ABD$).

$^1$Eucl. XII. 10.
(2) Through $B$, $D$ draw $V BW$, $XDY$ parallel to $AC$; and imagine a cylinder which has $AC$ for axis and the circles on $VX$, $WY$ as diameters for bases.

Then cylinder $VY = 2$ (cylinder $VD$) = 6 (cone $ABD$) = $\frac{3}{2}$ (sphere), from above.

From this theorem, to the effect that a sphere is four times as great as the cone with a great circle of the sphere as base and with height equal to the radius of the sphere, I conceived the notion that the surface of any sphere is four times as great as a great circle in it; for, judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius.

**Proposition** (Method, Proposition 7). *Any segment of a sphere has to the cone with the same base and height the ratio which the sum of the radius of the sphere and the height of the complementary segment has to the height of the complementary segment.*

segment $BAD : \text{cone } ABD = \frac{1}{2} AC + GC : GC$. 
Archimedes’ calculation of the volume of a sphere
Chapter 28

Surface area of sphere

**Proposition** (SCI 33). The surface of any sphere is equal to four times the greatest circle in it.

**Lower bounds of surface of sphere**

**Proposition** (SCI 21). A regular polygon of an even number of sides being inscribed in a circle, as ABC...A'...C'B'A, so that AA' is a diameter, if two angular points next but one to each other, as BB', be joined, and the other lines parallel to BB' and joining pairs of angular points be drawn, as CC', DD'...,

then

\[(BB' + CC' + \cdots) : AA' = A'B : BA.\]
Proposition (SCI 22). If a polygon be inscribed in a segment of a circle \( LAL' \), so that all its sides excluding the base are equal and their number is even, as \( LK \ldots A \ldots K'L' \), \( A \) being the middle point of the segment, and if the lines \( BB' \), \( CC' \ldots \), parallel to the base \( LL' \) and joining pairs of angular points be drawn, then

\[
(BB' + CC' + \ldots + LM) : AM = A'B : BA.
\]

Proposition (SCI 23). The surface of the sphere is greater than the surface described by the revolution of the polygon inscribed in the great circle about the diameter of the great circle.

Proposition (SCI 24). If a regular polygon \( AB \ldots A' \ldots B'A \), the number of whose sides is a multiple of 4, be inscribed in a great circle of a sphere, and if \( BB' \) subtending two sides be joined, and all the other lines parallel to \( BB' \) ad joining pairs of angular points be drawn, then the surface of the figure inscribed in the sphere by the revolution of the polygon about the diameter \( AA' \) is equal to a circle the square of whose radius is equal to the rectangle

\[
BA(BB' + CC' + \ldots).
\]

Proposition (SCI 25). The surface of the figure inscribed in a sphere as in the last propositions, consisting of portions of conical surfaces, is less than 4 times the greatest circle in the sphere.

Upper bounds of surface of sphere

Let a regular polygon, whose sides are a multiple of 4 in number, be circumscribed about a great circle of given sphere, as \( AB \ldots A' \ldots B'A \), and about the polygon, describe another circle, which will therefore have the same centre as the great circle of the sphere.
**Proposition (SCI 28).** The surface of the figure circumscribed to the given sphere is greater than that of the sphere itself.

**Proposition (SCI 29).** The surface of the figure circumscribed to the given sphere is equal to a circle the square on whose radius is equal to

\[ AB(BB' + CC' + \cdots) \]

**Proposition (SCI 30).** The surface of the figure circumscribed to the given sphere is greater than 4 times the great circle of the sphere.

**Proposition (SCI 32).** If a regular polygon with \(4n\) sides be inscribed in a great circle of a sphere, as \(ab\ldots a'b'a\), and a similar polygon \(AB\ldots A'B'\ldots A\) be described about the great circle, and if the polygons revolve with the great circle about the diameters \(aa', AA'\) respectively, so that they describe the surfaces of solid figures inscribed in and circumscribed to the sphere respectively, then the surfaces of the circumscribed and the inscribed figures are to one another in the duplicate ratio of their sides.

**Proof of SCI 33**

Let \(C\) be a circle equal to four times the great circle.

Then if \(C\) is not equal to the surface of the sphere, it must either be less or greater.

1. Suppose \(C\) less than the surface of the sphere.

It is then possible to find two lines \(\beta > \gamma\) such that

\[ \beta : \gamma < \text{surface of sphere} : C. \]

Let \(\delta\) be a mean proportional between \(\beta\) and \(\gamma\).
Suppose similar regular polygons with $4n$ sides circumscribed about and inscribed in a great circle such that the ratio of their sides is less than the ratio $\beta : \delta$. Let the polygons with the circle revolve together about a diameter common to all, describing solids of revolution as before. Then,

$$\frac{\text{surface of outer solid}}{\text{surface of inner solid}} = \frac{\text{side of outer polygon}^2}{\text{side of inner polygon}^2} < \frac{\beta^2}{\delta^2} = \frac{\beta}{\gamma} < \frac{\text{surface of sphere}}{C}.$$

But this is impossible, since the surface of the circumscribed solid is greater than that of the sphere, while the surface of the inscribed solid is less than $C$. Therefore $C$ is not less than the surface of the sphere.

II. Suppose $C$ greater than the surface of the sphere. Take lines $\beta$ and $\gamma$ such that

$$\beta : \gamma < C : \text{surface of sphere}.$$

Circumscribe and inscribe to the great circle similar regular polygons, as before, such that the ratio of their sides is less than the ratio $\beta : \delta$ ($\delta$ being a mean proportional of $\beta$ and $\gamma$), and suppose solids of revolution generated in the usual manner. Then in this case,

$$\frac{\text{surface of outer solid}}{\text{surface of inner solid}} = \frac{\text{side of outer polygon}^2}{\text{side of inner polygon}^2} < \frac{\beta^2}{\delta^2} = \frac{\beta}{\gamma} < \frac{C}{\text{surface of sphere}}.$$

But this is impossible, since the surface of the circumscribed solid is greater than $C$, while the surface of the inscribed solid is less than that of the sphere.

Thus $C$ is not greater than the surface of the sphere.

Therefore, since it is neither greater nor less, $C$ is equal to the surface of the sphere.

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$^1$Proposition 3: Given two unequal magnitudes and a circle, it is possible to inscribe a polygon in the circle and to describe another about it so that the side of the circumscribed polygon may have to the side of the inscribed polygon a ratio less than that of the greater magnitude to the less.
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