Fibonacci’s Book of Squares

0.14 Examples from The Book of Squares

Leonardo Pisano (1170 – 1240), also known as Fibonacci, treated in his Book of Squares, the square numbers as sum of consecutive odd numbers:

\[ n^2 = 1 + 3 + 5 + \cdots + (2n - 1). \]

Proof. (Without words)

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0.14.1 Proposition 14

Find an integer which, added to and subtracted from a square integer, yields always a square integer.

Solution. If \( x^2 + c = y^2 \) and \( y^2 + c = z^2 \), then \( x^2, y^2, z^2 \) form an arithmetic progression with common difference \( c \). This is equivalent to finding two adjacent chains of consecutive odd numbers with the same sum:

\[ 2x + 1, 2x + 3, \ldots, 2y - 1, \quad y - x \text{ terms with sum } c; \]
\[ 2y + 1, 2y + 3, \ldots, 2z - 1, \quad z - y \text{ terms with sum } c. \]

Fibonacci started with two integers \( m > n \), and constructed two adjacent chains of consecutive odd numbers as follows.

Assume that \( m \) and \( n \) are both odd. For the first chain, if \( \frac{m}{n} < \frac{m + n}{m - n} \), take \( m(m - n) \) consecutive odd numbers symmetrically about \( n(m + n) \);
if \( \frac{m}{n} > \frac{m + n}{m - n} \), take \( n(m + n) \) consecutive odd numbers symmetrically about \( m(m - n) \).

For the second chain, take \( n(m - n) \) consecutive odd numbers symmetrically about \( m(m + n) \).

These two chains always have equal sums.

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\[ \text{The sum of } 2k \text{ consecutive odd numbers placed symmetrically about an even number } 2a \text{ is } (2k)(2a) = 4ka. \]
If \( m \) and \( n \) have different parity, double the values of the “center number” and the number of terms involved above.

**Examples**

(1). \( m = 3, n = 1 \): 4 consecutive odd numbers centered at 6, and 2 consecutive odd numbers centered at 12.

\[
3 + 5 + 7 + 9 = 24 = 11 + 13; \quad 1^2 + 24 = 5^2, \quad 5^2 + 24 = 7^2.
\]

(2). \( m = 7, n = 3 \): 28 consecutive odd numbers centered at 30, and 12 consecutive odd numbers centered at 70.

\[
3 + 5 + \ldots + 57 = 840 = 59 + 61 + \ldots + 81; \quad 1^2 + 840 = 29^2, \quad 29^2 + 840 = 41^2.
\]

(3). \( m = 5, n = 4 \): 10 consecutive odd numbers centered at 80, and 8 consecutive odd numbers centered at 90.

\[
63 + 65 + \ldots + 81 = 720 = 83 + 85 + \ldots + 97; \quad 31^2 + 720 = 41^2, \quad 41^2 + 720 = 49^2.
\]

### 0.14.2 Proposition 17

Find a square rational number which, increased or diminished by 5, always yields a square number.

**Solution.** With \( m = 5, n = 4 \) as in Example 3 above, one obtains \( 31^2 + 720 = 41^2 \) and \( 41^2 + 720 = 49^2 \). Since \( 720 = 5 \cdot 12^2 \), division by \( 12^2 \) yields

\[
\left( \frac{31}{12} \right)^2 + 5 = \left( \frac{41}{12} \right)^2; \quad \left( \frac{41}{12} \right)^2 + 5 = \left( \frac{49}{12} \right)^2.
\]

### 0.14.3 The congruent number problem

Determine all squarefree integers which are the area of right triangles with rational sides.

If \( a, b, c \) are the rational sides of a right triangle with area \( n \), hypotenuse \( c \), then

\[
\left( \frac{c}{2} \right)^2 + n = (a + b)^2; \quad \left( \frac{c}{2} \right)^2 - n = (a - b)^2.
\]

Multiplying these equations, we have

\[
\left( \frac{c}{2} \right)^4 - n^2 = (a^2 - b^2)^2; \quad \left( \frac{c}{2} \right)^6 - n^2 \left( \frac{c}{2} \right)^2 = \frac{1}{4} c^2(a^2 - b^2)^2.
\]

Then, \( y^2 = x^3 - n^2x \) has a rational point \( \left( \frac{c^2}{4}, \frac{1}{2} c(a^2 - b^2) \right) \).

**Theorem 0.9.** \( n \) is a congruent number if and only if the elliptic curve \( y^2 = x^3 - n^2x \) has a rational point different from \((0,0), (\pm n, 0)\).