Chapter 12

Isotomic and isogonal conjugates

12.1 Isotomic conjugates

The Gergonne and Nagel points are examples of isotomic conjugates. Two points \(P\) and \(Q\) (not on any of the side lines of the reference triangle) are said to be isotomic conjugates if their respective traces are symmetric with respect to the midpoints of the corresponding sides. Thus,

\[ BX = X'C, \quad CY = Y'A, \quad AZ = Z'B. \]

We shall denote the isotomic conjugate of \(P\) by \(P^*\). If \(P = (x : y : z)\), then

\[ P^* = \left( \frac{1}{x} : \frac{1}{y} : \frac{1}{z} \right) = (yz : zx : xy). \]
12.1.1 The Gergonne and Nagel points

\[ G_e = \left( \frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c} \right), \quad N_a = (s - a : s - b : s - c). \]
12.1.2 The isotomic conjugate of the orthocenter

The isotomic conjugate of the orthocenter is the point
\[
H^* = (b^2 + c^2 - a^2 : c^2 + a^2 - b^2 : a^2 + b^2 - c^2).
\]

Its traces are the pedals of the deLongchamps point \(L_o\), the reflection of \(H\) in \(O\).

Exercise

1. Let \(XYZ\) be the cevian triangle of \(H^*\). Show that the lines joining \(X, Y, Z\) to the midpoints of the corresponding altitudes are concurrent. What is the common point? \(^1\)

2. Show that \(H^*\) is the perspector of the triangle of reflections of the centroid \(G\) in the sidelines of the medial triangle.

\(^1\)The centroid. Apply Menelaus theorem to triangle \(AA_H D\), and transversal \(MX\), where \(M\) is the midpoint of the altitude \(AA_H\).
12.1.3 Congruent-parallelians point

Given triangle $ABC$, we want to construct a point $P$ the three lines through which parallel to the sides cut out equal intercepts. Let $P = xA + yB + zC$ in absolute barycentric coordinates. The parallel to $BC$ cuts out an intercept of length $(1 - x)a$. It follows that the three intercepts parallel to the sides are equal if and only if

$$1 - x : 1 - y : 1 - z = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}.$$ 

The right hand side clearly gives the homogeneous barycentric coordinates of $I^*$, the isotomic conjugate of the incenter $I$. This is a point we can easily construct. Now, translating into absolute barycentric coordinates:

$$I^* = \frac{1}{2}[(1 - x)A + (1 - y)B + (1 - z)C] = \frac{1}{2}(3G - P).$$

we obtain $P = 3G - 2I^*$, and can be easily constructed as the point dividing the segment $I^*G$ externally in the ratio $I^*P : PG = 3 : -2$. The point $P$ is called the congruent-parallelians point of triangle $ABC$.

![Diagram of triangle ABC with points P, G, and I*]

Exercise

1. Calculate the homogeneous barycentric coordinates of the congruent-parallelian point and the length of the equal parallelians.

2. Let $A'B'C'$ be the midway triangle of a point $P$. The line $B'C'$ intersects $CA$ at

$$B_a = B'C' \cap CA, \quad C_a = B'C' \cap AB,$$

$$C_b = C'A' \cap AB, \quad A_b = C'A' \cap BC,$$

$$A_c = A'B' \cap BC, \quad B_c = A'B' \cap CA.$$

Determine $P$ for which the three segments $B_aC_a$, $C_bA_b$ and $A_cB_c$ have equal lengths.

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$^2$The isotomic conjugate of the incenter appears in ETC as the point $X_{75}$.

$^3$This point appears in ETC as the point $X_{192}$.

$^4$The common length of the equal parallelians is $\frac{2abc}{ab+bc+ca}$. 

The notation $ca + ab - bc : ab + bc - ca : bc + ca - ab$ is used to denote the equality of the segments.
12.2 Isogonal conjugates

Let \( P \) be a point with homogeneous barycentric coordinates \((x : y : z)\).

Let \( X \) and \( X' \) be points on \( BC \) such that the lines \( AX \) and \( AX' \) are isogonal, i.e., \( \angle BAX = \angle X'AC \). If this common angle is \( \theta \), then
\[
\frac{BX}{XC} = \frac{c}{b} \cdot \frac{\sin \theta}{\sin (A - \theta)} \quad \text{and} \quad \frac{BX'}{X'C} = \frac{c}{b} \cdot \frac{\sin (A - \theta)}{\sin \theta}.
\]
From this \( \frac{BX}{XC} : \frac{BX'}{X'C} = \frac{c^2}{b^2} \).

Since \( X = (0 : y : z) \), \( \frac{BX}{XC} = \frac{z}{y} \), we have \( \frac{BX'}{X'C} = \frac{c^2}{b^2} \cdot \frac{z}{y} = \frac{2}{b^2} \). The point \( X' \) has coordinates \((0 : \frac{b^2}{y} : \frac{c^2}{z})\).

Similarly, the reflections of the cevians \( BP \) and \( CP \) in the respective angle bisectors intersect \( CA \) at \( Y' \) and \( AB \) at \( Z' \) at \( \left(\frac{a^2}{x} : 0 : \frac{c^2}{z}\right) \) and \( (\frac{a^2}{x} : \frac{b^2}{y} : 0) \). The points \( X', Y', Z' \) are the traces of
\[
P^* = \left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}\right) = (a^2yz : b^2zx : c^2xy).
\]
The point \( P^* \) is called the isogonal conjugate of \( P \). Clearly, \( P \) is the isogonal conjugate of \( P^* \).

Example. It is easy to see that the incenter is its own isogonal conjugate, so are the excenters.

12.3 Examples of isogonal conjugates

12.3.1 The circumcenter and orthocenter

The circumcenter \( O \) and the orthocenter \( H \) are isogonal conjugates, since
\[
\angle (AO, AC) = \frac{\pi}{2} - \beta = -\angle (AH, AB),
\]
and similarly \( \angle (BO, BA) = -\angle (BH, BC) \), \( \angle (CO, CB) = -\angle (CH, CA) \).

Since \( OO^a = AH \), \( AOO^aH \) is a parallelogram, and \( HO^a = AO \). This means that the circle of reflections of \( O \) is congruent to the circumcircle. Therefore, the circle of reflections of \( H \) is the circumcircle, and the reflections of \( H \) lie on the circumcircle.
12.3.2 The isogonal conjugates of the Gergonne and Nagel points

Proposition 12.1. (a) The isogonal conjugate of the Gergonne point is the insimilicenter of the circumcircle and the incircle: $G_e^* = T_+$.  
(b) The isogonal conjugate of the Nagel point is the exsimilicenter of the circumcircle and the incircle: $N_a^* = T_-$. 
12.4 The symmedian point

The isogonal conjugate of the centroid $G = (1 : 1 : 1)$ is called the symmedian point, usually denoted by $K$. It is the point of concurrency of the symmedians, which are the isogonal lines of the medians. It has homogeneous barycentric coordinates $(a^2 : b^2 : c^2)$.

**Theorem 12.1.** The symmedian point is the perspector of the tangential triangle.

**Proof.** Let $A'B'C'$ be the tangential triangle, so that $B'C', C'A', A'B'$ are the tangents to the circumcircle at the vertices $A, B, C$ respectively. Extend $AB$ and $AC$ to $Z$ and $Y$ such that $A'B = A'Z$ and $A'C = A'Y$. We claim that $Y, A', Z$ are collinear. Note that

\[
\angle ZA'B = 180^\circ - 2\angle A'BZ = 180^\circ - 2\angle B'BA = 180^\circ - 2C,
\]

\[
\angle B'A'C = 180^\circ - 2\angle A'BC = 180^\circ - 2A,
\]

\[
\angle C'A'Y = 180^\circ - 2\angle A'CY = 180^\circ - 2\angle ACC' = 180^\circ - 2B.
\]

Hence,

\[
\angle ZA'B + \angle B'A'C + \angle C'A'Y = 540^\circ - 2(A + B + C) = 180^\circ,
\]

and $Y, A', Z$ are collinear. It follows that

(i) $AA'$ is a median of triangle $AYZ$, and

(ii) $AYZ$ and $ABC$ are similar.

If $D$ is the midpoint of $BC$, then the triangles $AA'Z$ and $ADC$ are similar. Therefore, $AA'$ and $AD$ are isogonal lines with respect to $AB$ and $AC$. Similarly, the $BB'$ and $CC'$ are isogonal to $B$- and $C$-medians. The lines $AA', BB', CC'$ therefore intersect at the isogonal conjugate of the centroid $G$, which is the point $K$. 

\[\square\]
12.4.1 The first Lemoine circle

Given triangle \(ABC\), how can one choose a point \(P\) so that when parallel lines are constructed through it to intersect each sideline at two points, the resulting six points are on a circle?

Suppose \(P = (u : v : w)\) in homogeneous barycentric coordinates.

1. \(AY_a : AC = AZ_a : AB = v + w : u + v + w\), so that \(AY_a = \frac{b(v+w)}{u+v+w}\) and \(AZ_a = \frac{c(v+w)}{u+v+w}\).
2. \(AY_c : AC = w : u + v + w\), so that \(AY_c = \frac{bw}{u+v+w}\).
3. \(AZ_b : AB = v : u + v + w\), so that \(AZ_b = \frac{cv}{u+v+w}\).

Since \(Y_c, Y_a, Z_a, Z_b\) are concyclic, by the intersection chords theorem,

\[
\frac{AY_a \cdot AY_c}{u+v+w} \cdot \frac{bw}{u+v+w} = \frac{AZ_a \cdot AZ_b}{u+v+w} \cdot \frac{cv}{u+v+w}.
\]

Therefore, \(v : w = b^2 : c^2\). Similarly, since \(X_b, X_c, Y_c, Y_a\) are concyclic, \(u : v = a^2 : b^2\).

From this we conclude if the six points are concyclic, \(P\) must be the symmedian point \(K = (a^2 : b^2 : c^2)\).

Conversely, if \(P\) is the symmedian point, then there are circles

(i) \(\mathcal{C}_a\) passing through \(Y_c, Y_a, Z_a, Z_b\),
(ii) \(\mathcal{C}_b\) passing through \(Z_a, Z_b, X_b, X_c\),
(iii) \(\mathcal{C}_c\) passing through \(X_b, X_c, Y_c, Y_a\).

This is an impossibility, since the radical axes of three circles are either parallel or concurrent. Therefore, the six points are concyclic, and the circle containing them is called the first Lemoine circle.
12.5 Triangles bounded by lines parallel to the sidelines

Theorem 12.2 (Homothetic center theorem). If parallel lines $X_bX_c$, $Y_cY_a$, $Z_aZ_b$ to the sides $BC$, $CA$, $AB$ of triangle $ABC$ are constructed such that

\[
AB : BX_c = AC : CX_b = 1 : t_1,
\]
\[
BC : CY_a = BA : AY_c = 1 : t_2,
\]
\[
CA : AZ_b = CB : BZ_a = 1 : t_3,
\]

these lines bound a triangle $A^*B^*C^*$ homothetic to $ABC$ with homothety ratio $1 + t_1 + t_2 + t_3$. The homothetic center is a point $P$ with homogeneous barycentric coordinates $t_1 : t_2 : t_3$.

**Proof.** Let $P$ be the intersection of $B^*B$ and $C^*C$. Since

\[
B^*C^* = B^*X_c + X_cX_b + X_bC^* = t_3a + (1 + t_1)a + t_2a = (1 + t_1 + t_2 + t_3)a,
\]

we have

\[
PB : PB^* = PC : PC^* = 1 : 1 + t_1 + t_2 + t_3.
\]

A similar calculation shows that $AA^*$ and $BB^*$ intersect at the same point $P$. This shows that $A^*B^*C^*$ is the image of $ABC$ under the homothety $h(P, 1 + t_1 + t_2 + t_3)$.

Now we compare areas. Note that

1. $\Delta(BZ_aX_c) = \frac{BX_c}{AB} \cdot \frac{BZ_a}{CB} \cdot \Delta(ABC) = t_1t_3\Delta(ABC)$,
2. $\frac{\Delta(PBC)}{\Delta(BZ_aB^*)} = \frac{PB}{BB^*} \cdot \frac{CB}{BZ_a} = \frac{1}{t_1 + t_2 + t_3} \cdot \frac{1}{t_3} = \frac{1}{t_3(t_1 + t_2 + t_3)}$.

Since $\Delta(BZ_aB^*) = \Delta(BZ_aX_c)$, we have $\Delta(PBC) = \frac{t_1}{t_1 + t_2 + t_3} \cdot \Delta(ABC)$.

Similarly, $\Delta(PCA) = \frac{t_2}{t_1 + t_2 + t_3} \cdot \Delta(ABC)$ and $\Delta(PAB) = \frac{t_3}{t_1 + t_2 + t_3} \cdot \Delta(ABC)$. It follows that

\[
\Delta(PBC) : \Delta(PCA) : \Delta(PAB) = t_1 : t_2 : t_3.
\]

$\square$
12.5.1 The symmedian point

Consider the square erected externally on the side $BC$ of triangle $ABC$, the line containing the outer edge of the square is the image of $BC$ under the homothety $h(A, 1 + t_1)$, where $1 + t_1 = \frac{\frac{a}{2\Delta} + a}{\frac{a}{2\Delta}} = 1 + \frac{a^2}{2\Delta}$, i.e., $t_1 = \frac{a^2}{2\Delta}$. Similarly, if we erect squares externally on the other two sides, the outer edges of these squares are on the lines which are the images of $CA, AB$ under the homotheties $h(B, 1 + t_2)$ and $h(C, 1 + t_3)$ with $t_2 = \frac{b^2}{2\Delta}$ and $t_3 = \frac{c^2}{2\Delta}$.

The triangle bounded by the lines containing these outer edges is called the Grebe triangle of $ABC$. It is homothetic to $ABC$ at
\[ \left( \frac{a^2}{2\Delta}, \frac{b^2}{2\Delta}, \frac{c^2}{2\Delta} \right) = \left( a^2 : b^2 : c^2 \right), \]
the symmedian point $K$, and the ratio of homothety is
\[ 1 + (t_1 + t_2 + t_3) = \frac{2\Delta + a^2 + b^2 + c^2}{2\Delta}. \]

Remark. Note that the homothetic center remains unchanged if we replaced $t_1, t_2, t_3$ by $kt_1, kt_2, kt_3$ for the same nonzero $k$. This means that if similar rectangles are constructed on the sides of triangle $ABC$, the lines containing their outer edges always bound a triangle with homothetic center $K$. 