Chapter 2

The shoemaker’s knife

2.1 The shoemaker’s knife

Let $P$ be a point on a segment $AB$. The region bounded by the three semicircles (on the same side of $AB$) with diameters $AB$, $AP$ and $PB$ is called a shoemaker’s knife. Suppose the smaller semicircles have radii $a$ and $b$ respectively. Let $Q$ be the intersection of the largest semicircle with the perpendicular through $P$ to $AB$. This perpendicular is an internal common tangent of the smaller semicircles.

Exercise

1. Show that the area of the shoemaker’s knife is $\pi ab$.

2. Let $UV$ be the external common tangent of the smaller semicircles, and $R$ the intersection of $PQ$ and $UV$. Show that

   (i) $UV = PQ$;

   (ii) $UR = PR = VR = QR$. Hence, with $R$ as center, a circle can be drawn passing through $P$, $Q$, $U$, $V$. 
3. Show that the circle through $U$, $P$, $Q$, $V$ has the same area as the shoemaker’s knife.

### 2.2 Circles in the shoemaker’s knife

**Theorem 2.1** (Archimedes). *The two circles each tangent to $CP$, the largest semicircle $AB$ and one of the smaller semicircles have equal radii $t$, given by

$$t = \frac{ab}{a + b}.$$*

**Proof.** Consider the circle tangent to the semicircles $O(a + b)$, $O_1(a)$, and the line $PQ$. Denote by $t$ the radius of this circle. Calculating in two ways the height of the center of this circle above the line $AB$, we have

$$(a + b - t)^2 - (a - b - t)^2 = (a + t)^2 - (a - t)^2.$$  

From this,

$$t = \frac{ab}{a + b}.$$

The symmetry of this expression in $a$ and $b$ means that the circle tangent to $O(a + b)$, $O_2(b)$, and $PQ$ has the same radius $t$.  

**Theorem 2.2** (Archimedes). *The circle tangent to each of the three semicircles has radius given by

$$\rho = \frac{ab(a + b)}{a^2 + ab + b^2}.$$*
2.2 Circles in the shoemaker’s knife

Proof. Let $\angle COO_2 = \theta$. By the cosine formula, we have

\[
\begin{align*}
(a + \rho)^2 &= (a + b - \rho)^2 + b^2 + 2b(a + b - \rho) \cos \theta, \\
(b + \rho)^2 &= (a + b - \rho)^2 + a^2 - 2a(a + b - \rho) \cos \theta.
\end{align*}
\]

Eliminating $\theta$, we have

\[
a(a + \rho)^2 + b(b + \rho)^2 = (a + b)(a + b - \rho)^2 + ab^2 + ba^2.
\]

The coefficients of $\rho^2$ on both sides are clearly the same. This is a linear equation in $\rho$:

\[
a^3 + b^3 + 2(a^2 + b^2)\rho = (a + b)^3 + ab(a + b) - 2(a + b)^2 \rho,
\]

from which

\[
4(a^2 + ab + b^2)\rho = (a + b)^3 + ab(a + b) - (a^3 + b^3) = 4ab(a + b),
\]

and $\rho$ is as above.

\[\square\]

Theorem 2.3 (Leon Bankoff). If the incircle $C(\rho)$ of the shoemaker’s knife touches the smaller semicircles at $X$ and $Y$, then the circle through the points $P$, $X$, $Y$ has the same radius as the Archimedean circles.
Proof. The circle through $P$, $X$, $Y$ is clearly the incircle of the triangle $CO_1O_2$, since

$$CX = CY = \rho, \quad O_1X = O_1P = a, \quad O_2Y = O_2P = b.$$  

The semiperimeter of the triangle $CO_1O_2$ is

$$a + b + \rho = (a + b) + \frac{ab(a + b)}{a^2 + ab + b^2} = \frac{(a + b)^3}{a^2 + ab + b^2}.$$  

The inradius of the triangle is given by

$$\sqrt{\frac{ab\rho}{a + b + \rho}} = \sqrt{\frac{ab \cdot ab(a + b)}{(a + b)^3}} = \frac{ab}{a + b}.$$  

This is the same as $t$, the common radius of Archimedes’ twin circles.

Construction of incircle of shoemaker’s knife

Let $Q_1$ and $Q_2$ be the “highest” points of the semicircles $O_1(a)$ and $O_2(b)$ respectively. The intersection of $O_1Q_2$ and $O_2Q_1$ is a point $C_3$ “above” $P$, and $C_3P = \frac{ab}{a+b} = t$. This gives a very easy construction of Bankoff’s circle in Theorem 2.3 above. From this, we obtain the points $X$ and $Y$. The center of the incircle of the shoemaker’s knife is the intersection $C$ of the lines $O_1X$ and $O_2Y$. The incircle of the shoemaker’s knife is the circle $C(X)$. It touches the largest semicircle of the shoemaker at $Z$, the intersection of $OC$ with this semicircle.

Note that $C_3(P)$ is the Bankoff circle, which has the same radius as the Archimedean circles.
Exercise

1. Show that the area of triangle $CO_1O_2$ is $\frac{ab(a+b)^2}{a^2+ab+b^2}$.

2. Show that the center $C$ of the incircle of the shoemaker’s knife is at a distance $2\rho$ from the line $AB$.

3. Show that the area of the shoemaker’s knife to that of the heart (bounded by semicircles $O_1(a)$, $O_2(b)$ and the lower semicircle $O(a+b)$) is as $\rho$ to $a+b$.

4. Show that the points of contact of the incircle $C(\rho)$ with the semicircles can be located as follows: $Y, Z$ are the intersections with $Q_1(A)$, and $X, Z$ are the intersections with $Q_2(B)$.

2.3 Archimedean circles in the shoemaker’s knife

Let $UV$ be the external common tangent of the semicircles $O_1(a)$ and $O_2(b)$, which extends to a chord $HK$ of the semicircle $O(a+b)$. Let $C_4$ be the intersection of $O_1V$ and $O_2U$. Since

$$O_1U = a, \quad O_2V = b, \quad \text{and} \quad O_1P : PO_2 = a : b,$$

$C_4P = \frac{ab}{a+b} = t$. This means that the circle $C_4(t)$ passes through $P$ and touches the common tangent $HK$ of the semicircles at $N$.

Let $M$ be the midpoint of the chord $HK$. Since $O$ and $P$ are symmetric (isotomic conjugates) with respect to $O_1O_2$,

$$OM + PN = O_1U + O_2V = a + b.$$
it follows that \((a + b) - QM = PN = 2t\). From this, the circle tangent to \(HK\) and the minor arc \(HK\) of \(O(a + b)\) has radius \(t\). This circle touches the minor arc at the point \(Q\).

**Theorem 2.4** (Thomas Schoch). The incircle of the curvilinear triangle bounded by the semicircle \(O(a + b)\) and the circles \(A(2a)\) and \(B(2b)\) has radius \(t = \frac{ab}{a+b}\).

**Proof.** Denote this circle by \(S(x)\). Note that \(SO\) is a median of the triangle \(SO_1O_2\). By Apollonius theorem,

\[
(2a + x)^2 + (2b + x)^2 = 2[(a + b)^2 + (a + b - x)^2].
\]

From this,

\[
x = \frac{ab}{a + b} = t.
\]

\[\square\]

**Exercise**

1. The circles \((C_1)\) and \((C'_1)\) are each tangent to the outer semicircle of the shoemaker’s knife, and to \(OQ_1\) at \(Q_1\); similarly for the circles \((C_2)\) and \((C'_2)\). Show that they have equal radii \(t = \frac{ab}{a+b}\).
2. We call the semicircle with diameter \( O_1O_2 \) the *midway semicircle* of the shoemaker’s knife.

Show that the circle tangent to the line \( PQ \) and with center at the intersection of \((O_1)\) and the midway semicircle has radius \( t = \frac{ab}{a+b} \).

3. Show that the radius of the circle tangent to the midway semicircle, the outer semicircle, and with center on the line \( PQ \) has radius \( t = \frac{ab}{a+b} \).