21.1 Menelaus’ theorem

**Theorem 21.1** (Menelaus). Given a triangle $ABC$ with points $X$, $Y$, $Z$ on the side lines $BC$, $CA$, $AB$ respectively, the points $X$, $Y$, $Z$ are collinear if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$$

**Proof.** ($\implies$) Let $W$ be the point on $AC$ such that $BW//XY$. Then,

$$\frac{BX}{XC} = \frac{WY}{YC}, \quad \text{and} \quad \frac{AZ}{ZB} = \frac{AY}{YW}.$$

It follows that

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{WY}{YC} \cdot \frac{CY}{YA} \cdot \frac{AY}{YW} = \frac{CY}{YC} \cdot \frac{AY}{YA} \cdot \frac{WY}{YW} = -1.$$
Suppose the line joining \( X \) and \( Z \) intersects \( AC \) at \( Y' \). From above,
\[
\frac{BX}{XC} \cdot \frac{CY'}{Y'A} \cdot \frac{AZ}{ZB} = -1 = \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB}.
\]

It follows that
\[
\frac{CY'}{Y'A} = \frac{CY}{YA}.
\]
The points \( Y' \) and \( Y \) divide the segment \( CA \) in the same ratio. These must be the same point, and \( X, Y, Z \) are collinear.

**Example 21.1.** The external angle bisectors of a triangle intersect their opposite sides at three collinear points.

![Diagram](image)

**Proof.** If the external bisectors are \( AX', BY', CZ' \) with \( X', Y', Z' \) on \( BC, CA, AB \) respectively, then
\[
\frac{BX'}{X'C} = \frac{c}{b}, \quad \frac{CY'}{Y'A} = \frac{a}{c}, \quad \frac{AZ'}{Z'B} = \frac{b}{a}.
\]

It follows that \( \frac{BX'}{X'C} \cdot \frac{CY'}{Y'A} \cdot \frac{AZ'}{Z'B} = -1 \) and the points \( X', Y', Z' \) are collinear. \( \square \)
21.2 Ceva’s theorem

**Theorem 21.2 (Ceva).** Given a triangle ABC with points X, Y, Z on the side lines BC, CA, AB respectively, the lines AX, BY, CZ are concurrent if and only if

\[
\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = +1.
\]

**Proof.** \((\implies=)\) Suppose the lines AX, BY, CZ intersect at a point \(P\). Consider the line \(BPY\) cutting the sides of triangle \(CAX\). By Menelaus’ theorem,

\[
\frac{CY}{YA} \cdot \frac{AP}{PX} \cdot \frac{XB}{BC} = -1, \quad \text{or} \quad \frac{CY}{YA} \cdot \frac{PA}{XP} \cdot \frac{BX}{BC} = +1.
\]

Also, consider the line \(CPZ\) cutting the sides of triangle \(ABX\). By Menelaus’ theorem again,

\[
\frac{AZ}{ZB} \cdot \frac{BC}{CX} \cdot \frac{XP}{PA} = -1, \quad \text{or} \quad \frac{AZ}{ZB} \cdot \frac{BC}{XC} \cdot \frac{XP}{PA} = +1.
\]

Multiplying the two equations together, we have

\[
\frac{CY}{YA} \cdot \frac{AZ}{ZB} \cdot \frac{BX}{XC} = +1.
\]

\((\iff=)\) Exercise.
Example 21.2. (1) The centroid. If $D$, $E$, $F$ are the midpoints of the sides $BC$, $CA$, $AB$ of triangle $ABC$, then clearly

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$ 

The medians $AD$, $BE$, $CF$ are therefore concurrent. Their intersection is the centroid $G$ of the triangle.

Consider the line $BGE$ intersecting the sides of triangle $ADC$. By the Menelaus theorem,

$$-1 = \frac{AG}{GD} \cdot \frac{DB}{BC} \cdot \frac{CE}{EA} = \frac{AG}{GD} \cdot \frac{-1}{2} \cdot 1.$$

It follows that $AG : GD = 2 : 1$. The centroid of a triangle divides each median in the ratio $2:1$.

(2) The incenter. Let $X$, $Y$, $Z$ be points on $BC$, $CA$, $AB$ such that $AX$, $BY$, $CZ$ bisect angles $BAC$, $CBA$ and $ACB$ respectively. Then

$$\frac{AZ}{ZB} = \frac{b}{a}, \quad \frac{BX}{XC} = \frac{c}{b}, \quad \frac{CY}{YA} = \frac{a}{c}.$$ 

It follows that

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{b}{a} \cdot \frac{c}{b} \cdot \frac{a}{c} = +1,$$

and $AX$, $BY$, $CZ$ are concurrent. Their intersection is the incenter of the triangle.
(3) In triangle $ABC$, $A = \frac{5\pi}{8}$, $B = \frac{\pi}{4}$, and $C = \frac{\pi}{8}$. Prove that the $A$-altitude, the $B$-bisector, and the $C$-median are concurrent.

**Solution.** Suppose $AX = 1$. Consider the two squares $AXBP$ and $AXTY$.

Note that triangle $ATC$ is isosceles since $\angle TAC = \angle ATX - \angle ACB = \frac{\pi}{4} - \frac{\pi}{8} = \frac{\pi}{8} = C$. Therefore,

\[
\frac{BX}{XC} = \frac{1}{1 + \sqrt{2}}, \quad \frac{CY}{YA} = \frac{BC}{BA} = \frac{2 + \sqrt{2}}{\sqrt{2}} = 1 + \sqrt{2}.
\]

Since $Z$ is the midpoint of $AB$,

\[
\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.
\]

The three lines $AX$, $BY$, $CZ$ are concurrent by Ceva’s theorem.
Exercise

1. Given triangle $ABC$ with $a = 15$, $b = 14$, $c = 9$.
   (a) Find points $X$ on $BC$, $Y$ on $CA$, and $Z$ on $AB$ such that $BX = CY = AZ$ and $AX$, $BY$, $CZ$ are concurrent.
   (b) Find also points $X'$ on $BC$, $Y'$ on $CA$, and $Z'$ on $AB$ such that $X'C = Y'A = Z'B$ and $AX'$, $BY'$, $CZ'$ are concurrent.

2. $ABC$ is a right triangle. Show that the lines $AX$, $BY$, and $CQ$ are concurrent.
3. \( ABC \) is a triangle with \( BC = 12 \), \( CA = 13 \), and \( AB = 15 \). Show that the median \( AD \), angle bisector \( BE \), and the altitude \( CF \) are concurrent.

4. Given three circles with centers \( A \), \( B \), \( C \) and distinct radii, show that the exsimilicenters of the three pairs of circles are collinear.
Menelaus and Ceva theorems
22.1 Barycentric coordinates

In a given triangle $ABC$, every point $P$ is coordinatized by a triple of
numbers $(x : y : z)$ in such a way that the system of masses $x$ at $A$, $y$
at $B$, and $z$ at $C$ will have its balance point at $P$. A mass $y$ at $B$ and
mass $z$ at $C$ will balance at the point $X$ on the line $BC$. A mass $x$ at $A$
and a mass $y + z$ at $X$ will balance at the point $P$.

\[
(y + z)X = yB + zC, \\
(x + y + z)P = xA + (y + z)X = xA + yB + zC.
\]

We say that with reference to triangle $ABC$, the point $P$ has
(i) absolute barycentric coordinate \( \frac{xA + yB + zC}{x + y + z} \) and
(ii) homogeneous barycentric coordinates $(x : y : z)$.  

![Diagram of barycentric coordinates](image_url)
22.2 Cevian and traces

Let $P$ be a point with homogeneous barycentric coordinates $(x : y : z)$ with reference to triangle $ABC$. The three lines joining a point $P$ to the vertices of the reference triangle $ABC$ are called the cevians of $P$. The intersections $X, Y, Z$ of these cevians with the side lines are called the traces of $P$. The coordinates of the traces can be very easily written down:

$$X = (0 : y : z), \quad Y = (x : 0 : z), \quad Z = (x : y : 0).$$

**Theorem 22.1** (Ceva theorem). *Three points $X, Y, Z$ on $BC, CA, AB$ respectively are the traces of a point if and only if they have coordinates of the form*

$$X = 0 : y : z, \quad Y = x : 0 : z, \quad Z = x : y : 0,$$

*for some $x, y, z$.***
Example 22.1. The centroid. The midpoint points of the sides have coordinates

\[
X = (0 : 1 : 1), \\
Y = (1 : 0 : 1), \\
Z = (1 : 1 : 0).
\]

The centroid \( G \) has coordinates \( (1 : 1 : 1) \).

Example 22.2. The incenter

The traces of the incenter have coordinates

\[
X = (0 : b : c), \\
Y = (a : 0 : c), \\
Z = (a : b : 0).
\]

The incenter \( I \) has coordinates \( (a : b : c) \).
22.3 Area and barycentric coordinates

Theorem 22.2. If in homogeneous barycentric coordinates with reference to triangle $ABC$, $P = (x : y : z)$, then

$$\Delta PBC : \Delta APC : \Delta ABP = x : y : z.$$ 

Proof. Consider the trace $X$ of $P$ on $BC$. Since a mass $x$ at $A$ and a mass $y + z$ at $X$ balance at $P$, $AP : PX = y + z : x$, and $PX : AX = x : x + y + z$. It follows that

$$\Delta PBC : \Delta ABC = PX : AX = x : x + y + z.$$ 

Similarly, $\Delta APC : \Delta ABC = y : x + y + z$ and $\Delta ABP : \Delta ABC = z : x + y + z$. Combining these, we have

$$x : y : z = \Delta PBC : \Delta APC : \Delta ABP.$$ 

Because of this theorem, homogeneous barycentric coordinates are also known as areal coordinates.
A useful area formula

If for \( i = 1, 2, 3 \), \( P_i = x_i \cdot A + y_i \cdot B + z_i \cdot C \) (in absolute barycentric coordinates), then the area of the oriented triangle \( P_1 P_2 P_3 \) is

\[
\Delta P_1 P_2 P_3 = \begin{vmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  x_3 & y_3 & z_3 \\
\end{vmatrix} \cdot \Delta ABC.
\]

**Example 22.3.** Let \( X, Y, Z \) be points on \( BC, CA, AB \) such that \( BX : XC = 2 : 1 \), \( CY : YA = 5 : 3 \), \( AZ : ZB = 3 : 2 \). (The numbers indicated along the lines are proportions of lengths, and are not actual lengths).

Their homogeneous barycentric coordinates are \( X = 0 : 1 : 2 \), \( Y = 3 : 0 : 5 \), and \( Z = 2 : 3 : 0 \).

\[
\Delta XYZ = \begin{vmatrix}
  1 & 1 & 1 \\
  3 & 0 & 5 \\
  2 & 3 & 0 \\
\end{vmatrix} \Delta ABC = \frac{28}{120} \Delta ABC = \frac{7}{30} \Delta ABC.
\]

\[
\Delta ABC = \frac{28}{120} \Delta ABC = \frac{7}{30} \Delta ABC.
\]
Example 22.4. With the same points $X, Y, Z$ in the preceding example, the lines $AX, BY, CZ$ bound a triangle $PQR$. Suppose triangle $ABC$ has area $\Delta$. Find the area of triangle $PQR$.

We have already known the coordinates of $X, Y, Z$. From these, it is easy to find those of $P, Q, R$:

<table>
<thead>
<tr>
<th>$P = BY \cap CZ$</th>
<th>$Q = CZ \cap AX$</th>
<th>$R = AX \cap BY$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y = (5 : 0 : 3)$</td>
<td>$Z = (2 : 3 : 0)$</td>
<td>$X = (0 : 1 : 2)$</td>
</tr>
<tr>
<td>$Z = (2 : 3 : 0)$</td>
<td>$X = (0 : 1 : 2)$</td>
<td>$Y = (5 : 0 : 3)$</td>
</tr>
</tbody>
</table>

This means that the absolute barycentric coordinates of $X, Y, Z$ are

$$P = \frac{1}{31} (10A + 15B + 6C), \quad Q = \frac{1}{11} (2A + 3B + 6C), \quad R = \frac{1}{19} (10A + 3B + 6C).$$

The area of triangle $PQR$

$$= \frac{1}{31 \cdot 11 \cdot 19} \begin{vmatrix} 10 & 15 & 6 \\ 2 & 3 & 6 \\ 10 & 3 & 6 \end{vmatrix} \cdot \Delta = \frac{576}{6479} \Delta.$$
Exercise

1. $ABC$ is a triangle of area 1, and $A_1, A_2, B_1, B_2, C_1, C_2$ are the points of trisection of $BC, CA$ and $AB$ respectively. Which of the two areas is larger, the one bounded by three lines $AA_1, BB_1, CC_1$ or the one bounded by the four lines $BB_1, BB_2, CC_1, CC_2$?

2. Let $X, Y, Z$ be points dividing $BC, CA, AB$ in the golden ratio. Let $P$ be the intersection of $BY$ and $CZ$, $Q$ that $CZ$ and $AX$, and $R$ that of $AX$ and $BY$. Show that
   (i) $P, Q, R$ are respectively the midpoints of $BY, CZ, AX$;
   (ii) $P$ divides $QC$ in the golden ratio; so do $Q$ and $R$ divide $RA$ and $PB$.
   (iii) Compare the areas of triangles $XYZ$ and $PQR$. ¹
