Chapter 24

The lemniscate

Given two points $A$ and $B$ at a distance $2a$ apart, the lemniscate is the locus of point $P$ such that $AP \cdot BP = a^2$.

If $(x, y)$ is a point on the locus, then

$((x - a)^2 + y^2)((x + a)^2 + y^2) = (a^2)^2$.

Rewriting this as

$(x^2 + y^2 + a^2 - 2ax)(x^2 + y^2 + a^2 + 2ax) = a^4$.

In terms of polar coordinates, this is

$(\rho^2 + a^2 - 2a \rho \cos \theta)(\rho^2 + a^2 + 2a \rho \cos \theta) = a^4$. 
Cancelling a common term \( a^4 \) and then a common factor \( \rho^2 \), we have
\[
\rho^2 + 2a^2 - 4a^2 \cos^2 \theta = 0,
\]
or
\[
\rho^2 = 2a^2(2 \cos^2 \theta - 1) = 2a^2 \cos 2\theta.
\]

**Remark.** The rectangular equation is
\[
(x^2 + y^2)^2 - 2a^2(x^2 - y^2) = 0.
\]

**Exercise**

1. Calculate the area enclosed by the lemniscate.
2. Formulate the integral for the perimeter of the lemniscate.
Supplement 20A: Generation of the ellipse and hyperbola

Consider a fixed point $F(c, 0)$ and a variable point $P$ on the circle $x^2 + y^2 = a^2$. We find the envelope of the perpendicular to $FP$ at $P$. If $P$ has coordinates $(a \cos \theta, a \sin \theta)$, then this line has equation $F(\theta) = 0$, where

$$
F(\theta) = (x - a \cos \theta)(c - a \cos \theta) + (-a \sin \theta)(y - a \sin \theta) \\
= (c - a \cos \theta)x - a \sin \theta \cdot y + a(a - c \cos \theta) \\
= -a(x + c) \cos \theta - ay \cdot \sin \theta + (a^2 + cx).
$$

To find the envelope of the line, we eliminate $\theta$ from $F(\theta) = 0$ and $F'(\theta) = 0$.

$$
a(x + c) \cos \theta + ay \cdot \sin \theta = (a^2 + cx), \\
-a(x + c) \sin \theta + ay \cdot \cos \theta = 0.
$$

Note that the sum of the squares of the expressions on the left hand side is independent of $\theta$:

$$
a^2(x + c)^2 + a^2y^2 = (a^2 + cx)^2.
$$

Collecting terms, we have

$$
(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2),
$$
or

$$
\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.
$$

Therefore, this is an ellipse if $c^2 < a^2$ and a hyperbola if $c^2 > a^2$. The point $F$ is a focus.

If $c = a$, then $F$ is on the given circle and all such perpendiculars pass through the antipodal point of $F$. The envelope degenerate into this single point.
Supplement 20B: More examples of envelopes

Example 0.1. Consider the lines whose intercepts in the first quadrant has a fixed length $a$. If this line makes an angle $t$ with the (negative) $x$ axis, it has equation $\frac{x}{a \cos t} + \frac{y}{a \sin t} = 1$, or
\[
f(x, y, t) := x \sin t + y \cos t - a \sin t \cos t = 0.
\]
Here, $f_t(x, y, t) = x \cos t - y \sin t - a(\cos^2 t - \sin^2 t)$. Solving $f(x, y, t) = 0$ and $f_t(x, y, t) = 0$ simultaneously for $x$ and $y$, we have
\[
x = a \cos^3 t, \quad y = a \sin^3 t.
\]
The envelope is the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

![Figure 1: The astroid](image)

Example 0.2. Consider the family of lines whose intercepts on the $x$- and $y$-axes have a fixed total length $a$. If the $x$-intercept has length $t \leq a$, then the line has equation $\frac{x}{t} + \frac{y}{a-t} = 1$, or
\[
f(x, y, t) := (a-t)x + ty - t(a-t) = 0.
\]
Here, $f_t(x, y, t) = -x + y - (a-2t)$. Solving $f = 0$ and $f_t = 0$, we have
\[
x = \frac{t^2}{a}, \quad y = \frac{(a-t)^2}{a}.
\]
This is the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$. Its focus is the point $\left(\frac{a}{2}, \frac{a}{2}\right)$, and its directrix is the line $x + y = 0$. The distances from $\left(\frac{t^2}{a}, \frac{(a-t)^2}{a}\right)$ to the focus and the directrix are both $\frac{2t^2 - 2at + a^2}{\sqrt{2a}}$. 
Example 0.3. Consider the circle with center \((at^2, 2at)\) on the parabola \(y^2 = 4ax\) and passing through the origin. It is given by

\[ f(x, y, t) = x^2 + y^2 - 2at^2x - 4aty. \]

Here, \(f_t(x, y, t) = -4a(tx + y)\). Solving \(f = 0\) and \(f_t = 0\), we have

\[ (x, y) = \left(\frac{-2at^2}{1 + t^2}, \frac{2at^3}{1 + t^2}\right). \]
Exercise

(1) Find the cartesian equation of the envelope.

(2) Consider the circle through the origin and with center \( (at, \ \frac{a}{t}) \) on the rectangular hyperbola \( xy = a^2 \).

(i) Find the equation of the circle.

(ii) Find the envelope of the circle as a parametrized curve.

(iii) Find the Cartesian equation of the envelope.

Example 0.4. Consider a point \( P(t) = (a \cos t, \ b \sin t) \) on the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). The equation of the circle with center \( P(t) \) and passing through the origin is

\[
x^2 + y^2 - 2a \cos t \cdot x - 2b \sin t \cdot y = 0
\]

To find the envelope, we eliminate \( t \) from

\[
\begin{align*}
ax \cos t + by \sin t &= \frac{1}{2} (x^2 + y^2), \\
-ax \sin t + by \cos t &= 0
\end{align*}
\]

by noting that the sum of the squares of the expressions on the left hand side is independent of \( t \):

\[
a^2 x^2 + b^2 y^2 = \frac{1}{4} (x^2 + y^2)^2.
\]

Exercise

Find the polar equation of the curve.
Supplement 21: Application to the duplication of the cube (correction)

Let $CD$ be the diameter perpendicular to $AB$, and $M$ the midpoint of $OC$. Join $B, M$ to intersect the cissoid at $Q$. Let $X$ be the intersection of the diameter $AB$ with the perpendicular from $Q$. Then $XB : XQ = OB : OM = 2 : 1$, and $XP'$ and $AX$ are the two mean proportions between them. Therefore, $\frac{XP'}{XB} \cdot OB$ and $\frac{AX}{AQ} \cdot OM$ are two mean proportions between $OB$ and $OM$. The latter is $\sqrt{2} \cdot OM$. 
Supplement 24: Rectangular linkages

Given two robot arms of length $b$ hinged at foci $A = \left(-\frac{a}{2}, 0\right)$ and $B = \left(\frac{a}{2}, 0\right)$, connect the free ends with a third arm of length $a$. This configuration is a rectangular linkage. Let $M$ be the midpoint of the middle robot arm. We determine the locus of $M$ as $P$ varies on the circle $(B)$.

Let $(\rho, \theta)$ be the polar coordinates of $M$ (relative to $O$ and the line $AB$). Note that $AQB\!P$ is a symmetric trapezoid. If the diagonals $PQ$ and $AB$ intersect at $K$, then $KO = KM$. Therefore, $\angle MKB = 2\angle MOB = 2\theta$. In triangle $BKP$, we have

$$KB = OB - OK = \frac{a}{2} - \frac{\rho}{2 \cos \theta},$$
$$KP = MP + KM = \frac{a}{2} + \frac{\rho}{2 \cos \theta}.$$

Applying the law of cosines, we have

$$b^2 = \left(\frac{a}{2} - \frac{\rho}{2 \cos \theta}\right)^2 + \left(\frac{a}{2} + \frac{\rho}{2 \cos \theta}\right)^2 - 2 \left(\frac{a}{2} - \frac{\rho}{2 \cos \theta}\right) \left(\frac{a}{2} + \frac{\rho}{2 \cos \theta}\right) \cos 2\theta$$
$$= \frac{1}{2} \left( a^2 + \frac{\rho^2}{\cos^2 \theta} \right) - \frac{1}{2} \left( a^2 - \frac{\rho^2}{\cos^2 \theta} \right) \cos 2\theta$$
$$= \frac{1}{2} a^2(1 - \cos 2\theta) + \frac{1}{2} \cdot \frac{\rho^2}{\cos^2 \theta} (1 + \cos 2\theta)$$
$$= a^2 \sin^2 \theta + \rho^2.$$

Therefore, the polar equation of the middle point $M$ of the middle robot arm is

$$\rho^2 = b^2 - a^2 \sin^2 \theta.$$
defined for $\theta$ satisfying $|\sin \theta| \leq \frac{b}{a}$. If $b = \sqrt{2}a$, this is the lemniscate 
\[ \rho^2 = 2a^2 \cos 2\theta. \]

Here is a more general result. Since the vector $QP = a(\cos 2\theta, \sin 2\theta)$, we can easily determine, for a fixed $t \in [0, 1]$, the locus of the point $P_t$ dividing $PQ$ in the ratio $1-t:t$.

\[
P_t = M + \frac{1}{2}(2t-1)QP = \sqrt{b^2 - a^2 \sin^2 \theta} (\cos \theta, \sin \theta) + \frac{1}{2}(2t-1) (a \cos 2\theta, a \sin 2\theta) = \left( \frac{a}{2} (1-2t), 0 \right) + (\sqrt{b^2 - a^2 \sin^2 \theta} + a(2t-1) \cos \theta)(\cos \theta, \sin \theta).
\]

If we shift the pole to the point $\left( \frac{a}{2} (1-2t), 0 \right)$, we have the polar curve 
\[ \rho = \sqrt{b^2 - a^2 \sin^2 \theta} + a(2t-1) \cos \theta \]
defined for $\theta$ satisfying $|\sin \theta| \leq \frac{b}{a}$.

Figure 3: $t = \frac{1}{2}$
The lemniscate

Figure 4: $t = \frac{4}{5}$

Figure 5: $t = \frac{1}{3}$

Figure 6: $t = \frac{3}{5}$

Figure 7: $t = \frac{2}{5}$

Figure 8: $t = \frac{1}{10}$
Supplement 24B: The lemniscate integral

The integral for the perimeter of the lemniscate is an example of an elliptic integral.

**Theorem 0.2** (Gauss). Let $a \leq b$ be given positive real numbers.

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{\pi}{\text{agM}(a, b)}.$$

**Proof.** We first show that the integral is invariant if $a$ and $b$ are replaced by $\sqrt{ab}$ and $\frac{a+b}{2}$ respectively.

Consider the substitution $x = \frac{1}{2} \left( y - \frac{ab}{y} \right)$. As $y$ ranges from 0 to $\infty$, $x$ ranges from $-\infty$ to $\infty$. Note that

$$x^2 + ab = \frac{1}{4} \left( y - \frac{ab}{y} \right)^2 + ab = \frac{1}{4} \left( y + \frac{ab}{y} \right)^2 = \frac{(y^2 + ab)^2}{4y^2},$$

$$x^2 + \left( \frac{a+b}{2} \right)^2 = \frac{1}{4} \left( y - \frac{ab}{y} \right)^2 + \left( \frac{a+b}{2} \right)^2 = \frac{(y^2 + a^2)(y^2 + b^2)}{4y^2},$$

$$dx = \frac{1}{2} \left( 1 + \frac{ab}{y^2} \right) dy = \frac{(y^2 + ab)dy}{2y^2}.$$

It follows that

$$\int_{x=-\infty}^{x=\infty} \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = \int_{y=0}^{y=\infty} \frac{2dy}{\sqrt{(y^2 + a^2)(y^2 + b^2)}} = \int_{x=-\infty}^{x=\infty} \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}}.$$

If we continue replacing by geometric and arithmetic means, we obtain a sequence of nested intervals

$$[a, b] \supset [a_1, b] \supset [a_2, b] \supset \cdots \supset [a_n, b_n] \supset \cdots$$

converging to $\text{agM}(a, b)$. Moreover,

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2 + a_n^2)(x^2 + b_n^2)}} = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}}$$

for every $n$. Since

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + b_n^2} \leq \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2 + a_n^2)(x^2 + b_n^2)}} \leq \int_{-\infty}^{\infty} \frac{dx}{x^2 + a_n^2}.$$
we have
\[ \frac{\pi}{b_n} \leq \int_{-\infty}^{\infty} \frac{2dy}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} \leq \frac{\pi}{a_n} \]
for every \( n \). Since \((a_n)\) and \((b_n)\) both converge to \( \text{agM}(a, b) \), we have
\[ \frac{\pi}{\text{agM}(a, b)} \leq \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} \leq \frac{\pi}{\text{agM}(a, b)} \]
and
\[ \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{\pi}{\text{agM}(a, b)}. \]

\[ \square \]

**Theorem 0.3.** The perimeter of the lemniscate \( \rho^2 = 2a^2 \cos 2\theta \) is \( \frac{4\sqrt{2}\pi a}{\text{agM}(1, \sqrt{2})} \).

**Proof.** The integral for the perimeter of the lemniscate, namely, \( \int_0^1 \frac{dt}{\sqrt{1-t^4}} \), can be transformed into one of the form considered in the theorem above, by putting
\[ t = \frac{1}{\sqrt{x^2 + 1}}. \]
Note that \( x \) ranges from 0 to \( \infty \) oppositely as \( t \) ranges from 0 to 1, and
\[ dt = \frac{-xdx}{(x^2 + 1)^{3/2}}, \]
\[ 1 - t^4 = 1 - \frac{1}{(x^2 + 1)^2} = \frac{x^2(x^2 + 2)}{(x^2 + 1)^2}. \]
Therefore, the perimeter of the lemniscate is
\[ 4\sqrt{2}a \int_{t=0}^{t=1} \frac{dt}{\sqrt{1-t^4}} = 4\sqrt{2}a \int_{x=0}^{x=\infty} \frac{dx}{\sqrt{(x^2 + 1)(x^2 + 2)}} = \frac{2\sqrt{2}\pi a}{\text{agM}(1, \sqrt{2})}. \]

\[ \square \]