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Chapter 1

Some Basic Theorems

1.1 The Pythagorean Theorem

Theorem 1.1 (Pythagoras). The lengths $a \leq b < c$ of the sides of a right triangle satisfy the relation

$$a^2 + b^2 = c^2.$$

Proof.

Theorem 1.2 (Converse of Pythagoras’ theorem). If the lengths of the sides of $\triangle ABC$ satisfy $a^2 + b^2 = c^2$, then the triangle has a right angle at $C$.

Proof. Consider a right triangle $XYZ$ with $\angle Z = 90^\circ$, $YZ = a$, and $XZ = b$. By the Pythagorean theorem, $XY^2 = YZ^2 + XZ^2 = a^2 + b^2 = c^2 = AB^2$. It follows that $XY = AB$, and $\triangle ABC \equiv \triangle XYZ$ by the SSS test, and $\angle C = \angle Z = 90^\circ$.  

\[\square\]
Example 1.1. Trigonometric ratios of $30^\circ$, $45^\circ$, and $60^\circ$:

\[
\begin{array}{|c|c|c|c|}
\hline
\theta & \sin \theta & \cos \theta & \tan \theta \\
\hline
30^\circ & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{3} \\
45^\circ & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 \\
60^\circ & \frac{\sqrt{3}}{2} & \frac{1}{2} & \sqrt{3} \\
\hline
\end{array}
\]

Example 1.2. For an arbitrary point $P$ on the minor arc $BC$ of the circumcircle of an equilateral triangle $ABC$, $AP = BP + CP$.

Proof. If $Q$ is the point on $AP$ such that $PQ = PC$, then the isosceles triangle $CPQ$ is equilateral since $\angle CPQ = \angle CBA = 60^\circ$. Note that $\angle ACQ = \angle BCP$. Thus, $\triangle ACQ \equiv \triangle BCP$ by the SAS congruence test. From this, $AQ = BP$, and $AP = AQ + QP = BP + CP$. \qed
Example 1.3. Given a rectangle $ABCD$, to construct points $P$ on $BC$ and $Q$ on $CD$ such that triangle $APQ$ is equilateral.

Let $BCY$ and $CDX$ are equilateral triangles inside the rectangle $ABCD$. Extend the lines $AX$ and $AY$ to intersect $BC$ and $CD$ respectively at $P$ and $Q$. $APQ$ is equilateral.

Proof. Suppose $AB = 2a$ and $BC = 2b$. The distance of $X$ above $AB = 2b - \sqrt{3}a$. By the Pythagorean theorem, $AX^2 = a^2 + (2b - \sqrt{3}a)^2 = 4(a^2 + b^2 - \sqrt{3}ab)$. $X$ is the midpoint of $AP$. Therefore, $AP^2 = 16(a^2 + b^2 - \sqrt{3}ab)$. Similarly, $AQ^2 = (2AY)^2 = 4AY^2 = 4 \cdot (b^2 + (2a - \sqrt{3}b)^2) = 16(a^2 + b^2 - \sqrt{3}ab)$. Finally, $CP = 2b - 2(2b - \sqrt{3}a) = 2(\sqrt{3}a - b)$, $CQ = 2(\sqrt{3}b - a)$, and $PQ^2 = CP^2 + CQ^2 = 4((\sqrt{3}a - b)^2 + (\sqrt{3}b - a)^2) = 16(a^2 + b^2 - \sqrt{3}ab)$. It follows that $AP = AQ = PQ$, and triangle $APQ$ is equilateral.
1.2 Constructions of geometric mean

We present two ruler-and-compass constructions of the geometric means of two quantities given as lengths of segments. These are based on Euclid’s proof of the Pythagorean theorem.

Construct the altitude at the right angle to meet $AB$ at $P$ and the opposite side $ZZ'$ of the square $ABZZ'$ at $Q$. Note that the area of the rectangle $AZQP$ is twice of the area of triangle $AZC$. By rotating this triangle about $A$ through a right angle, we obtain the congruent triangle $ABY'$, whose area is half of the area of the square on $AC$. It follows that the area of rectangle $AZQP$ is equal to the area of the square on $AC$. For the same reason, the area of rectangle $BZ'QP$ is equal to that of the square on $BC$. From these, the area of the square on $AB$ is equal to the sum of the areas of the squares on $BC$ and $CA$.

Construction. Given two segments of length $a < b$, mark three points $P, A, B$ on a line such that $PA = a$, $PB = b$, and $A, B$ are on the same side of $P$. Describe a semicircle with $PB$ as diameter, and let the perpendicular through $A$ intersect the semicircle at $Q$. Then $PQ^2 = PA \cdot PB$, so that the length of $PQ$ is the geometric mean of $a$ and $b$. 
Construction. Given two segments of length $a, b$, mark three points $A, P, B$ on a line ($P$ between $A$ and $B$) such that $AP = a, PB = b$. Describe a semicircle with $AB$ as diameter, and let the perpendicular through $P$ intersect the semicircle at $Q$. Then $PQ^2 = PA \cdot PB$, so that the length of $PQ$ is the geometric mean of $a$ and $b$. 
1.3 The golden ratio

Given a segment $AB$, a point $P$ in the segment is said to divide it in the golden ratio if $AP^2 = PB \cdot AB$. Equivalently, $\frac{AP}{PB} = \frac{\sqrt{5}+1}{2}$. We shall denote this golden ratio by $\varphi$. It is the positive root of the quadratic equation $x^2 = x + 1$.

Construction (Division of a segment in the golden ratio). Given a segment $AB$,
(1) draw a right triangle $ABM$ with $BM$ perpendicular to $AB$ and half in length,
(2) mark a point $Q$ on the hypotenuse $AM$ such that $MQ = MB$,
(3) mark a point $P$ on the segment $AB$ such that $AP = AQ$.

Then $P$ divides $AB$ into the golden ratio.

Suppose $PB$ has unit length. The length $\varphi$ of $AP$ satisfies

$$\varphi^2 = \varphi + 1.$$

This equation can be rearranged as

$$\left( \varphi - \frac{1}{2} \right)^2 = \frac{5}{4}.$$

Since $\varphi > 1$, we have

$$\varphi = \frac{1}{2} \left( \sqrt{5} + 1 \right).$$

Note that

$$\frac{AP}{AB} = \frac{\varphi}{\varphi + 1} = \frac{1}{\varphi} = \frac{2}{\sqrt{5} + 1} = \frac{\sqrt{5} - 1}{2}.$$

This explains the construction above.

1.3.1 The regular pentagon

Consider a regular pentagon $ACBDE$. It is clear that the five diagonals all have equal lengths. Note that
(1) $\angle ACB = 108^\circ$, 
1.3 The golden ratio

(2) triangle $CAB$ is isosceles, and
(3) $\angle CAB = \angle CBA = (180^\circ - 108^\circ) \div 2 = 36^\circ$.

In fact, each diagonal makes a $36^\circ$ angle with one side, and a $72^\circ$ angle with another.

It follows that
(4) triangle $PBC$ is isosceles with $\angle PBC = \angle PCB = 36^\circ$,
(5) $\angle BPC = 180^\circ - 2 \times 36^\circ = 108^\circ$, and
(6) triangles $CAB$ and $PBC$ are similar.

Note that triangle $ACP$ is also isosceles since
(7) $\angle ACP = \angle APC = 72^\circ$. This means that $AP = AC$.

Now, from the similarity of $CAB$ and $PBC$, we have $AB : AC = BC : PB$. In other words $AB \cdot AP = AP \cdot PB$, or $AP^2 = AB \cdot PB$. This means that $P$ divides $AB$ in the golden ratio.

Construction. Given a segment $AB$, we construct a regular pentagon $ACBDE$ with $AB$ as a diagonal.

(1) Divide $AB$ in the golden ratio at $P$.
(2) Construct the circles $A(P)$ and $P(B)$, and let $C$ be an intersection of these two circles.
(3) Construct the circles $A(AB)$ and $B(C)$ to intersect at a point $D$ on the same side of $BC$ as $A$.
(4) Construct the circles $A(P)$ and $D(P)$ to intersect at $E$.

Then $ACBDE$ is a regular pentagon with $AB$ as a diagonal.
1.4 Basic construction principles

1.4.1 Perpendicular bisector locus

A variable point $P$ is equidistant from two fixed points $A$ and $B$ if and only if $P$ lies on the perpendicular bisector of the segment $AB$.

The perpendicular bisectors of the three sides of a triangle are concurrent at the circumcenter of the triangle. This is the center of the circumcircle, the circle passing through the three vertices of the triangle.

![Diagram showing perpendicular bisectors and circumcircle]

The circumcenter of a right triangle is the midpoint of its hypotenuse.
1.4.2 Angle bisector locus

A variable point $P$ is equidistant from two fixed lines $\ell$ and $\ell'$ if and only if $P$ lies on the bisector of one of the angles between $\ell$ and $\ell'$.

The bisectors of the three angles of a triangle are concurrent at a point which is at equal distances from the three sides. With this point as center, a circle can be constructed tangent to the sides of the triangle. This is the incircle of the triangle. The center is the incenter.

**Proof.** Let the bisectors of angles $B$ and $C$ intersect at $I$. Consider the pedals of $I$ on the three sides. Since $I$ is on the bisector of angle $B$, $IX = IZ$. Since $I$ is also on the bisector of angle $C$, $IX = IY$. It follows $IX = IY = IZ$, and the circle, center $I$, constructed through $X$, also passes through $Y$ and $Z$, and is tangent to the three sides of the triangle. \qed
1.4.3 Tangency of circles

Two circles \( (O) \) and \( (O') \) are tangent to each other if they are tangent to a line \( \ell \) at the same line \( P \), which is a common point of the circles. The tangency is internal or external according as the circles are on the same or different sides of the common tangent \( \ell \).

![Diagram of tangency of circles]

The line joining their centers passes through the point of tangency.

The distance between their centers is the sum or difference of their radii, according as the tangency is external or internal.

1.4.4 Construction of tangents of a circle

A tangent to a circle is a line which intersects the circle at only one point. Given a circle \( O(A) \), the tangent to a circle at \( A \) is the perpendicular to the radius \( OA \) at \( A \).

![Diagram of construction of tangents]

If \( P \) is a point outside a circle \( (O) \), there are two lines through \( P \) tangent to the circle. Construct the circle with \( OP \) as diameter to intersect \( (O) \) at two points. These are the points of tangency.

The two tangents have equal lengths since the triangles \( OAP \) and \( OBP \) are congruent by the RHS test.
Example 1.4. Given two congruent circles each with center on the other circle, to construct a circle tangent to the center line, and also to the given circles, one internally and the other externally.

Let $AB = a$. Suppose the required circle has radius $r$, and $AT = x$, where $T$ is the point of tangency $T$ with the center line.

$$(a + x)^2 + r^2 = (a + r)^2,$$

$$x^2 + r^2 = (a - r)^2.$$ 

From these, we have

$$x + a \frac{a}{2} = \frac{\sqrt{3}}{2} \cdot a,$$

$$\frac{a}{2} + x = 2r.$$

This means that if $M$ is the midpoint of $AB$, then $MT = \frac{\sqrt{3}}{2} \cdot a$, which is the height of the equilateral triangle on $AB$. In other words, if $C$ is an intersection of the two given circles, then $CM$ and $MT$ are two adjacent sides of a square. Furthermore, the side opposite to $CM$ is a diameter of the required circle!
1.5 The intersecting chords theorem

**Theorem 1.3.** Given a point $P$ and a circle $O(r)$, if a line through $P$ intersects the circle at two points $A$ and $B$, then $PA \cdot PB = OP^2 - r^2$, independent of the line.

![Diagram](https://via.placeholder.com/150)

*Proof.* Let $M$ be the midpoint of $AB$. Note that $OM$ is perpendicular to $AB$. If $P$ is outside the circle, then

$$PA \cdot PB = (PM + MA)(PM - MB)$$
$$= (PM + MA)(PM - MA)$$
$$= PM^2 - MA^2$$
$$= (OM^2 + PM^2) - (OM^2 + MA^2)$$
$$= OP^2 - r^2.$$  

The same calculation applies to the case when $P$ is inside or on the circle, provided that the lengths of the directed segments are signed.

The quantity $OP^2 - r^2$ is called the **power** of $P$ with respect to the circle. It is positive, zero, or negative according as $P$ is outside, on, or inside the circle.

**Corollary 1.4** (Intersecting chords theorem). If two chords $AB$ and $CD$ of a circle intersect, extended if necessary, at a point $P$, then $PA \cdot PB = PC \cdot PD$.

In particular, if the tangent at $T$ intersects $AB$ at $P$, then $PA \cdot PB = PT^2$.

![Diagram](https://via.placeholder.com/150)

The converse of the intersecting chords theorem is also true.
Theorem 1.5. Given four points $A, B, C, D$, if the lines $AB$ and $CD$ intersect at a point $P$ such that $PA \cdot PB = PC \cdot PD$ (as signed products), then $A, B, C, D$ are concyclic.

In particular, if $P$ is a point on a line $AB$, and $T$ is a point outside the line $AB$ such that $PA \cdot PB = PT^2$, then $PT$ is tangent to the circle through $A, B, T$.

Example 1.5. Let $ABC$ be a triangle with $B = 2C$. Then $b^2 = c(c + a)$.

Construct a parallel through $C$ to the bisector $BE$, to intersect the extension of $AB$ at $F$. Then

$$\angle AFC = \angle ABE = \frac{1}{2} \cdot \angle ABC = \angle ACB.$$ 

This means that $AC$ is tangent to the circle through $B, C, F$. By the intersecting chord theorem, $AC^2 = AB \cdot AF$, i.e., $b^2 = c(c + a)$. 
1.6 Ptolemy’s theorem

**Theorem 1.6 (Ptolemy).** A convex quadrilateral $ABCD$ is cyclic if and only if

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$ 

**Proof.** (Necessity) Assume, without loss of generality, that $\angle BAD > \angle ABD$. Choose a point $P$ on the diagonal $BD$ such that $\angle BAP = \angle CAD$. Triangles $BAP$ and $CAD$ are similar, since $\angle ABP = \angle ACD$. It follows that $AB : AC = BP : CD$, and

$$AB \cdot CD = AC \cdot BP.$$ 

Now, triangles $ABC$ and $APD$ are also similar, since $\angle BAC = \angle BAP + \angle PAC = \angle DAC + \angle PAC = \angle PAD$, and $\angle ACB = \angle ADP$. It follows that $AC : BC = AD : PD$, and

$$BC \cdot AD = AC \cdot PD.$$ 

Combining the two equations, we have

$$AB \cdot CD + BC \cdot AD = AC(BP + PD) = AC \cdot BD.$$ 

(Sufficiency). Let $ABCD$ be a quadrilateral satisfying (**). Locate a point $P'$ such that $\angle BAP' = \angle CAD$ and $\angle ABP' = \angle ACD$. Then the triangles $ABP$ and $ACD$ are similar. It follows that

$$AB : AP' : BP' = AC : AD : CD.$$ 

From this we conclude that

(i) $AB \cdot CD = AC \cdot BP'$, and

(ii) triangles $ABC$ and $AP'D$ are similar since $\angle BAC = \angle P'AD$ and $AB : AC = AP' : AD$.

Consequently, $AC : BC = AD : P'D$, and

$$AD \cdot BC = AC \cdot P'D.$$ 

Combining the two equations,

$$AC(BP' + P'D) = AB \cdot CD + AD \cdot BC = AC \cdot BD.$$ 

It follows that $BP' + P'D = BD$, and the point $P'$ lies on diagonal $BD$. From this, $\angle ABD = \angle ABP' = \angle ACD$, and the points $A, B, C, D$ are concyclic. \qed
Chapter 2

The laws of sines and cosines

2.1 The law of sines

Theorem 2.1 (The law of sines). Let $R$ denote the circumradius of a triangle $ABC$.

$$2R = \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$ 

Since the area of a triangle is given by $\Delta = \frac{1}{2}bc \sin \alpha$, the circumradius can be written as

$$R = \frac{abc}{4\Delta}.$$
2.2 The orthocenter

Why are the three altitudes of a triangle concurrent?

Let $ABC$ be a given triangle. Through each vertex of the triangle we construct a line parallel to its opposite side. These three parallel lines bound a larger triangle $A'B'C'$. Note that $ABCB'$ and $ACBC'$ are both parallelograms since each has two pairs of parallel sides. It follows that $B'A = BC = AC'$ and $A$ is the midpoint of $B'C'$.

Consider the altitude $AX$ of triangle $ABC$. Seen in triangle $A'B'C'$, this line is the perpendicular bisector of $B'C'$ since it is perpendicular to $B'C'$ through its midpoint $A$. Similarly, the altitudes $BY$ and $CZ$ of triangle $ABC$ are perpendicular bisectors of $C'A'$ and $A'B'$. As such, the three lines $AX$, $BY$, $CZ$ concur at a point $H$. This is called the orthocenter of triangle $ABC$.

**Proposition 2.2.** The reflections of the orthocenter in the sidelines lie on the circumcircle.

**Proof.** It is enough to show that the reflection $H_a$ of $H$ in $BC$ lies on the circumcircle. Consider also the reflection $O_a$ of $O$ in $BC$. Since $AH$ and $OO_a$ are parallel and have the same length $(2R \cos \alpha)$, $AOO_aH$ is a parallelogram. On the other hand, $HOO_aH_a$ is a isosceles trapezoid. It follows that $OH_a = HO_a = AO$, and $H_a$ lies on the circumcircle. \(\square\)
2.3 The law of cosines

Given a triangle $ABC$, we denote by $a$, $b$, $c$ the lengths of the sides $BC$, $CA$, $AB$ respectively.

**Theorem 2.3** (The law of cosines).

\[
c^2 = a^2 + b^2 - 2ab \cos \gamma.
\]

**Proof.** Let $AX$ be the altitude on $BC$.

\[
c^2 = BX^2 + AX^2 = (a - b \cos \gamma)^2 + (b \sin \gamma)^2 = a^2 - 2ab \cos \gamma + b^2 (\cos^2 \gamma + \sin^2 \gamma) = a^2 + b^2 - 2ab \cos \gamma.
\]

---

**Theorem 2.4** (Stewart). Let $X$ be a point on the sideline $BC$ of triangle $ABC$.

\[
a \cdot AX^2 = BX \cdot b^2 + XC \cdot c^2 - a \cdot BX \cdot XC.
\]

Here, the lengths of the directed segments on the line $BC$ are signed. Equivalently, if $BX : XC = \lambda : \mu$, then

\[
AX^2 = \frac{\lambda b^2 + \mu c^2}{\lambda + \mu} - \frac{\lambda \mu a^2}{(\lambda + \mu)^2}.
\]

**Proof.** Use the cosine formula to compute the cosines of the angles $AXB$ and $AXC$, and note that $\cos AXC = -\cos AXB$. 

\[\square\]
Example 2.1. (Napoleon’s theorem). If similar isosceles triangles $XBC$, $YCA$ and $ZAB$ (of base angle $\theta$) are constructed externally on the sides of triangle $ABC$, the lengths of the segments $YZ$, $ZX$, $XZ$ can be computed easily. For example, in triangle $AYZ$, $AY = \frac{b}{2} \sec \theta$, $AZ = \frac{c}{2} \sec \theta$ and $\angle YAZ = \alpha + 2\theta$.

By the law of cosines,

$$YZ^2 = AY^2 + AZ^2 - 2AY \cdot AZ \cdot \cos YAZ$$

$$= \frac{\sec^2 \theta}{4} (b^2 + c^2 - 2bc \cos(\alpha + 2\theta))$$

$$= \frac{\sec^2 \theta}{4} (b^2 + c^2 - 2bc \cos \alpha \cos 2\theta + 2bc \sin \alpha \sin 2\theta)$$

$$= \frac{\sec^2 \theta}{4} (b^2 + c^2 - (b^2 + c^2 - a^2) \cos 2\theta + 4\Delta \sin 2\theta)$$

$$= \frac{\sec^2 \theta}{4} (a^2 \cos 2\theta + (b^2 + c^2)(1 - \cos 2\theta) + 4\Delta \sin 2\theta).$$

Likewise, we have

$$ZX^2 = \frac{\sec^2 \theta}{4} (b^2 \cos 2\theta + (c^2 + a^2)(1 - \cos 2\theta) + 4\Delta \sin 2\theta)$$

$$XY^2 = \frac{\sec^2 \theta}{4} (c^2 \cos 2\theta + (a^2 + b^2)(1 - \cos 2\theta) + 4\Delta \sin 2\theta).$$

It is easy to note that $YZ = ZX = XY$ if and only if $\cos 2\theta = \frac{1}{2}$, i.e., $\theta = 30^\circ$. In this case, the points $X$, $Y$, $Z$ are the centers of equilateral triangles erected externally on $BC$, $CA$, $AB$ respectively. The same conclusion holds if the equilateral triangles are constructed internally on the sides. This is the famous Napoleon theorem.

Theorem 2.5 (Napoleon). If equilateral triangles are constructed on the sides of a triangle, either all externally or all internally, then their centers are the vertices of an equilateral triangle.
Example 2.2. (Orthogonal circles) Given three points $A$, $B$, $C$ that form an acute-angled triangle, construct three circles with these points as centers that are mutually orthogonal to each other.

Let $BC = a$, $CA = b$, and $AB = c$. If these circles have radii $R_a$, $R_b$, $R_c$ respectively, then

$$R_b^2 + R_c^2 = a^2, \quad R_c^2 + R_a^2 = b^2, \quad R_a^2 + R_b^2 = c^2.$$  

From these,

$$R_a^2 = \frac{1}{2}(b^2 + c^2 - a^2), \quad R_b^2 = \frac{1}{2}(c^2 + a^2 - b^2), \quad R_c^2 = \frac{1}{2}(a^2 + b^2 - c^2).$$

These are all positive since $ABC$ is an acute triangle. Consider the perpendicular foot $E$ of $B$ on $AC$. Note that $AE = c \cos A$, so that $R_a^2 = \frac{1}{2}(b^2 + c^2 - a^2) = bc \cos A = AC \cdot AE$. It follows if we extend $BE$ to intersect at $Y$ the semicircle constructed externally on the side $AC$ as diameter, then, $AY^2 = AC \cdot AE = R_a^2$. Therefore we have the following simple construction of these circles.

(1) With each side as diameter, construct a semicircle externally of the triangle.
(2) Extend the altitudes of the triangle to intersect the semicircles on the same side. Label these $X$, $Y$, $Z$ on the semicircles on $BC$, $CA$, $AB$ respectively. These satisfy $AY = AZ$, $BZ = BX$, and $CX = CY$.
(3) The circles $A(Y)$, $B(Z)$ and $C(X)$ are mutually orthogonal to each other.
2.4 The centroid

Let $E$ and $F$ be the midpoints of $AC$ and $AB$ respectively, and $G$ the intersection of the medians $BE$ and $CF$.

Construct the parallel through $C$ to $BE$, and extend $AG$ to intersect $BC$ at $D$, and this parallel at $H$. 

By the converse of the midpoint theorem, $G$ is the midpoint of $AH$, and $HC = 2 \cdot GE$

Join $BH$. By the midpoint theorem, $BH // CF$. It follows that $BHCG$ is a parallelogram. Therefore, $D$ is the midpoint of (the diagonal) $BC$, and $AD$ is also a median of triangle $ABC$. We have shown that the three medians of triangle $ABC$ intersect at $G$, which we call the centroid of the triangle.

Furthermore,

$$AG = GH = 2GD,$$

$$BG = HC = 2GE,$$

$$CG = HB = 2GF.$$ 

The centroid $G$ divides each median in the ratio $2 : 1$.

**Theorem 2.6** (Apollonius). If $m_a$ denotes the length of the median on the side $BC$,

$$m_a^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2).$$

**Example 2.3.** Suppose the medians $BE$ and $CF$ of triangle $ABC$ are perpendicular. This means that $BC^2 + CG^2 = BC^2$, where $G$ is the centroid of the triangle. In terms of the lengths, we have $\frac{4}{9}m_b^2 + \frac{4}{9}m_c^2 = a^2$; $4(m_b^2 + m_c^2) = 9a^2$; $(2c^2 + 2a^2 - b^2) + (2a^2 + 2b^2 - c^2) = 9a^2$; $b^2 + c^2 = 5a^2$.

This relation is enough to describe, given points $B$ and $C$, the locus of $A$ for which the medians $BE$ and $CF$ of triangle $ABC$ are perpendicular. Here, however, is a very easy construction: From $b^2 + c^2 = 5a^2$, we have $m_a^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2) = \frac{3}{4}a^2$; $m_a = \frac{3}{2}a$. The locus of $A$ is the circle with center at the midpoint of $BC$, and radius $\frac{3}{2} \cdot BC$. 
2.5 The angle bisector theorem

Theorem 2.7 (Angle bisector theorem). The bisectors of an angle of a triangle divide its opposite side in the ratio of the remaining sides. If $AX$ and $AX'$ respectively the internal and external bisectors of angle $BAC$, then $BX : XC = c : b$ and $BX' : X'C = c : -b$.

Proof. Construct lines through $C$ parallel to the bisectors $AX$ and $AX'$ to intersect the line $AB$ at $Z$ and $Z'$.

(1) Note that $\angle AZC = \angle BAX = \angle XAC = \angle ACZ$. This means $AZ = AC$. Clearly, $BX : XC = BA : AZ = BA : AC = c : b$.

(2) Similarly, $AZ' = AC$, and $BX' : X'C = BA : AZ' = BA : -AC = c : -b$.

2.5.1 The lengths of the bisectors

Proposition 2.8. (a) The lengths of the internal and external bisectors of angle $A$ are respectively

$$t_a = \frac{2bc}{b + c} \cos \frac{\alpha}{2} \quad \text{and} \quad t'_a = \frac{2bc}{|b - c|} \sin \frac{\alpha}{2}.$$

Proof. Let $AX$ and $AX'$ be the bisectors of angle $A$.

(1) Consider the area of triangle $ABC$ as the sum of those of triangles $AXC$ and $ABX$. We have

$$\frac{1}{2} t_a (b + c) \sin \frac{\alpha}{2} = \frac{1}{2} bc \sin \alpha.$$
The laws of sines and cosines

From this,
\[ t_a = \frac{bc}{b+c} \cdot \sin \frac{\alpha}{2} = \frac{2bc}{b+c} \cdot \cos \frac{\alpha}{2}. \]

(2) Consider the area of triangle as the difference between those of \( ABX' \) and \( ACX' \).

Remarks. (1) \( \frac{2bc}{b+c} \) is the harmonic mean of \( b \) and \( c \). It can be constructed as follows. If the perpendicular to \( AX \) at \( X \) intersects \( AC \) and \( AB \) at \( Y \) and \( Z \), then \( AY = AZ = \frac{2bc}{b+c} \).

(2) Applying Stewart’s Theorem with \( \lambda = c \) and \( \mu = \pm b \), we also obtain the following expressions for the lengths of the angle bisectors:

\[ t_a^2 = bc \left( 1 - \left( \frac{a}{b+c} \right)^2 \right), \]
\[ t'_a^2 = bc \left( \left( \frac{a}{b-c} \right)^2 - 1 \right). \]

Example 2.4. (Steiner-Lehmus theorem). A triangle with two equal angle bisectors is isosceles. More precisely, if the bisectors of two angles of a triangle have equal lengths, then the two angles are equal.

Proof. We show that if \( a < b \), then \( t_a > t_b \). Note that from \( a < b \) we conclude

(i) \( \alpha < \beta \) and \( \cos \frac{\alpha}{2} > \cos \frac{\beta}{2} \);
(ii) \( bc > ac, b(c + a) > a(b + c) \);
\[ \frac{b}{b+c} > \frac{c}{c+a}, \frac{2bc}{b+c} > \frac{2ca}{c+a}. \]

From (i) and (ii), we have
\[ t_a = \frac{2bc}{b+c} \cdot \cos \frac{\alpha}{2} > \frac{2ca}{c+a} \cdot \cos \frac{\beta}{2} = t_b. \]

The same reasoning shows that \( a > b \Rightarrow t_a < t_b \). It follows that if \( t_a = t_b \), then \( a = b \).
2.6 The circle of Apollonius

Theorem 2.9. A and B are two fixed points. For a given positive number $k \neq 1$, the locus of points $P$ satisfying $AP : PB = k : 1$ is the circle with diameter $XY$, where $X$ and $Y$ are points on the line $AB$ such that $AX : XB = k : 1$ and $AY : YB = k : -1$.

Proof. Since $k \neq 1$, points $X$ and $Y$ can be found on the line $AB$ satisfying the above conditions.

Consider a point $P$ not on the line $AB$ with $AP : PB = k : 1$. Note that $PX$ and $PY$ are respectively the internal and external bisectors of angle $APB$. This means that angle $XPY$ is a right angle, and $P$ lies on the circle with $XY$ as diameter.

Conversely, let $P$ be a point on this circle. We show that $AP : BP = k : 1$. Let $B'$ be a point on the line $AB$ such that $PX$ bisects angle $APB'$. Since $PA$ and $PB$ are perpendicular to each other, the line $PB$ is the external bisector of angle $APB'$, and

$$\frac{AY}{YB'} = -\frac{AX}{XB'} = \frac{XA}{XB'} = \frac{AY -XA}{YX}.$$  

On the other hand,

$$\frac{AY}{YB} = -\frac{AX}{XB} = \frac{XA}{XB} = \frac{AY -XA}{YX}.$$  

Comparison of the two expressions shows that $B'$ coincides with $B$, and $PX$ is the bisector of angle $APB$. It follows that $\frac{PA}{PB} = \frac{AX}{XB} = k$. \qed

---

$^1$If $k = 1$, the locus is clearly the perpendicular bisector of the segment $AB$. 
The laws of sines and cosines
Chapter 3

The tritangent circles

3.1 The incircle

Let the incircle of triangle $ABC$ touch its sides $BC$, $CA$, $AB$ at $X$, $Y$, $Z$ respectively.

If $s$ denotes the semiperimeter of triangle $ABC$, then

$$AY = AZ = s - a,$$
$$BZ = BX = s - b,$$
$$CX = CY = s - c.$$ 

The inradius of triangle $ABC$ is the radius of its incircle. It is given by

$$r = \frac{2\Delta}{a + b + c} = \frac{\Delta}{s}.$$
Example 3.1. If triangle $ABC$ has a right angle at $C$, then the inradius $r = s - c$.

It follows that if $d$ is the diameter of the incircle, then $a + b = c + d$.

Example 3.2. An equilateral triangle of side $2a$ is partitioned symmetrically into a quadrilateral, an isosceles triangle, and two other congruent triangles. If the inradii of the quadrilateral and the isosceles triangle are equal, find this inradius.$^1$

Suppose each side of the equilateral triangle has length 2, each of the congruent circles has radius $r$, and $\angle ACX = \theta$.

(i) From triangle $AXC$, $r = \frac{2}{\cot 30^\circ + \cot \theta}$.

(ii) Note that $\angle BCY = \frac{1}{2}(60^\circ - 2\theta) = 30^\circ - \theta$. It follows that $r = \tan(30^\circ - \theta) = \frac{1}{\cot(30^\circ - \theta)} = \frac{\cot \theta - \cot 30^\circ}{\cot 30^\circ \cot \theta + 1}$.

By putting $\cot \theta = x$, we have $\frac{2}{\sqrt{3} + x} = \frac{x - \sqrt{3}}{\sqrt{3} x + 1}$; $x^2 - 3 = 2\sqrt{3}x + 2$; $x^2 - 2\sqrt{3}x - 5 = 0$, and $x = \sqrt{3} + 2\sqrt{2}$. (The negative root is rejected). From this, $r = \frac{2}{\sqrt{3} + x} = \frac{1}{\sqrt{3} + \sqrt{2}} = \sqrt{3} - \sqrt{2}$.

To construct the circles, it is enough to mark $Y$ on the altitude through $A$ such that $AY = \sqrt{3} - r = \sqrt{2}$. The construction is now evident.

$^1(\sqrt{3} - \sqrt{2})a.$
Example 3.3. (Conway’s circle). Given triangle $ABC$, extend
(i) $CA$ and $BA$ to $Y_a$ and $Z_a$ such that $AY_a = AZ_a = a,$
(ii) $AB$ and $CB$ to $Z_b$ and $X_b$ such that $BZ_b = BX_b = b,$
(iii) $BC$ and $AC$ to $X_c$ and $Y_c$ such that $CX_c = CY_c = c.$

The six points $X_b, X_c, Y_c, Y_a, Z_a, Z_b$ are concyclic. The circle containing them has center $I$ and radius $\sqrt{r^2 + s^2}$.

Proof. Let the incircle be tangent to $BC$ at $X$. $BX = s - b \implies X_bX = b + (s - b) = s.$
From this, $IX_b^2 = r^2 + s^2$. Similarly, each of the remaining five points is at the same distance from $I$. They lie on a circle, center $I$, radius $\sqrt{r^2 + s^2}$.
3.2 Euler’s formula

Theorem 3.1 (Euler’s formula). \( OI^2 = R^2 - 2Rr \)

\[ OI^2 = R^2 - 2Rr \]

\[ OI^2 = R^2 - 2Rr \]

Proof. Extend \( AI \) to intersect the circumcircle at \( M \). Since \( \angle BAM = \angle MAC = \frac{A}{2} \), \( M \) is the midpoint of the arc \( BC \) of the circumcircle. Note that

\[ \angle MBI = \angle MBC + \angle CBI = \angle MAC + \angle CBI = \angle BAI + \angle IBA = \angle BIM. \]

It follows that \( IM = BM \). By the law of sines, \( BM = 2R \sin \frac{A}{2} \). Since \( AI = \frac{r}{\sin \frac{A}{2}} \),

\[ AI \cdot IM = 2Rr. \]

This is equal to the power of \( I \) with respect to the circumcircle, namely, \( = R^2 - OI^2 \). Therefore, \( OI^2 = R^2 - 2Rr \).

\[ \square \]

Corollary 3.2. \( R \geq 2r; \) equality holds if and only if the triangle is equilateral.
3.3 Steiner’s porism

Given a triangle $ABC$ with incircle $(I)$ and circumcircle $O$, let $A'$ be an arbitrary point on circumcircle. Join $A'$ to $I$, to intersect the circumcircle again at $M'$, and let $A'Y', A'Z'$ be the tangents to the incircle. Construct a circle, center $M'$, through $I$ to intersect the circumcircle at $B'$ and $C'$.

Denote by $\theta$ between $A'I$ and each tangent.

It is known that $A'I \cdot IM' = 2Rr$ (power of incenter in $(O)$). Since $A'I = \frac{r}{\sin \theta}$, we have $IM' = 2R \sin \theta$. Therefore, $M'B' = M'C' = 2R \sin \theta$, and by the law of sines, $\angle B'A'M = \angle C'A'M = \theta$. It follows that $A'B'$ and $A'C'$ are tangent to the incircle $(I)$. Since $M'$ is the midpoint of the arc $B'C'$, the circle $M'(B')$ passes through the incenter of triangle $A'B'C'$. This means that $I$ is the incenter, and $(I)$ the incircle of triangle $A'B'C'$.
3.4 The excircles

The internal bisector of each angle and the external bisectors of the remaining two angles are concurrent at an excenter of the triangle. An excircle can be constructed with this as center, tangent to the lines containing the three sides of the triangle.

The exradii of a triangle with sides $a$, $b$, $c$ are given by

$$r_a = \frac{\Delta}{s-a}, \quad r_b = \frac{\Delta}{s-b}, \quad r_c = \frac{\Delta}{s-c}.$$  

The areas of the triangles $I_aBC$, $I_aCA$, and $I_aAB$ are $\frac{1}{2}ar_a$, $\frac{1}{2}br_a$, and $\frac{1}{2}cr_a$ respectively. Since

$$\Delta = -\Delta I_aBC + \Delta I_aCA + \Delta I_aAB,$$

we have

$$\Delta = \frac{1}{2}r_a(-a+b+c) = r_a(s-a),$$

from which $r_a = \frac{\Delta}{s-a}$. 
3.5 Heron’s formula for the area of a triangle

Consider a triangle $ABC$ with area $\Delta$. Denote by $r$ the inradius, and $r_a$ the radius of the excircle on the side $BC$ of triangle $ABC$. It is convenient to introduce the semiperimeter $s = \frac{1}{2}(a + b + c)$.

(1) From the similarity of triangles $AIY$ and $AI'Y'$,

$$
\frac{r}{r_a} = \frac{s - a}{s}.
$$

(2) From the similarity of triangles $CIY$ and $I'CY'$,

$$
r \cdot r_a = (s - b)(s - c).
$$

(3) From these,

$$
\begin{align*}
    r &= \sqrt{\frac{(s - a)(s - b)(s - c)}{s}}, \\
    r_a &= \sqrt{\frac{s(s - b)(s - c)}{s - a}}.
\end{align*}
$$

**Theorem 3.3** (Heron’s formula).

$$
\Delta = \sqrt{s(s - a)(s - b)(s - c)}.
$$

*Proof.* $\Delta = rs$. \qed

**Proposition 3.4.**

$$
\begin{align*}
    \tan \frac{\alpha}{2} &= \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}, \\
    \cos \frac{\alpha}{2} &= \sqrt{\frac{s(s - a)}{bc}}, \\
    \sin \frac{\alpha}{2} &= \sqrt{\frac{(s - b)(s - c)}{bc}}.
\end{align*}
$$
The tritangent circles
Chapter 4

The arbelos

4.1 Archimedes’ twin circles

Theorem 4.1 (Archimedes). The two circles each tangent to $CP$, the largest semicircle $AB$ and one of the smaller semicircles have equal radii $t$, given by

$$t = \frac{ab}{a+b}.$$ 

Proof. Consider the circle tangent to the semicircles $O(a+b)$, $O_1(a)$, and the line $PQ$. Denote by $t$ the radius of this circle. Calculating in two ways the height of the center of this circle above the line $AB$, we have

$$(a + b - t)^2 - (a - b - t)^2 = (a + t)^2 - (a - t)^2.$$ 

From this,

$$t = \frac{ab}{a+b}.$$ 

The symmetry of this expression in $a$ and $b$ means that the circle tangent to $O(a+b)$, $O_2(b)$, and $PQ$ has the same radius $t$. 

\qed
4.1.1 Harmonic mean and the equation $\frac{1}{a} + \frac{1}{b} = \frac{1}{t}$

The harmonic mean of two quantities $a$ and $b$ is $\frac{2ab}{a+b}$. In a trapezoid of parallel sides $a$ and $b$, the parallel through the intersection of the diagonals intercepts a segment whose length is the harmonic mean of $a$ and $b$. We shall write this harmonic mean as $2t$, so that $\frac{1}{a} + \frac{1}{b} = \frac{1}{t}$.

Here is another construction of $t$, making use of the formula for the length of an angle bisector in a triangle. If $BC = a$, $AC = b$, then the angle bisector $CZ$ has length $t_c = \frac{2ab}{a+b} \cos \frac{C}{2} = 2t \cos \frac{C}{2}$.

The length $t$ can therefore be constructed by completing the rhombus $CXZY$ (by constructing the perpendicular bisector of $CZ$ to intersect $BC$ at $X$ and $AC$ at $Y$). In particular, if the triangle contains a right angle, this trapezoid is a square.

4.1.2 Construction of the Archimedean twin circles

Construct the circle $P(C_3)$ to intersect the diameter $AB$ at $P_1$ and $P_2$ so that $P_1$ is on $AP$ and $P_2$ is on $PB$. The center $C_1$ respectively $C_2$ is the intersection of the circle $O_1(P_2)$ respectively $O_2(P_1)$ and the perpendicular to $AB$ at $P_1$ respectively $P_2$. 
4.2 The incircle

**Theorem 4.2 (Archimedes).** The incircle of the arbelos has radius

\[ \rho = \frac{r_1 r_2 (r_1 + r_2)}{r_1^2 + r_1 r_2 + r_2^2}. \]

**Proof.** Let \( \angle COO_3 = \theta \). By the law of cosines, we have

\[
\begin{align*}
(r_1 + \rho)^2 &= (r_1 + r_2 - \rho)^2 + r_2^2 + 2r_2(r_1 + r_2 - \rho) \cos \theta, \\
(r_2 + \rho)^2 &= (r_1 + r_2 - \rho)^2 + r_1^2 - 2r_1(r_1 + r_2 - \rho) \cos \theta.
\end{align*}
\]

Eliminating \( \theta \), we have

\[ r_1 (r_1 + \rho) + r_2 (r_2 + \rho) = (r_1 + r_2) (r_1 + r_2 - \rho)^2 + r_1 r_2^2 + r_2 r_1^2. \]

The coefficients of \( \rho^2 \) on both sides are clearly the same. This is a linear equation in \( \rho \):

\[ r_1^3 + r_2^3 + 2(r_1^2 + r_2^2) \rho = (r_1 + r_2)^3 + r_1 r_2 (r_1 + r_2) - 2(r_1 + r_2)^2 \rho, \]

from which

\[ 4(r_1^2 + r_1 r_2 + r_2^2) \rho = (r_1 + r_2)^3 + r_1 r_2 (r_1 + r_2) - (r_1^3 + r_2^3) = 4r_1 r_2 (r_1 + r_2), \]

and \( \rho \) is as above. \( \square \)
4.2.1 Construction of the incircle of the arbelos

In [?], Bankoff published the following remarkable theorem which gives a construction of the incircle of the arbelos of the incircle, much simpler than the one we designed before from Archimedes’ proof. The simplicity of the construction is due to the existence of a circle congruent to Archimedes’ twin circles.

**Theorem 4.3** (Bankoff). *The points of tangency of the incircle of the arbelos with the semicircles (AC) and (CB), together with C, are the points of tangency of the incircle (W₃) of triangle O₁O₂O₃ with the sides of the triangle. This circle (W₃) is congruent to Archimedes’ twin circles (W₁) and (W₂).*

Proof. Since O₁Q = O₁C, O₂C = O₂R, and O₃R = O₃Q, the points C, Q, R are the points of tangency of the incircle of triangle O₁O₂O₃ with its sides. The semi-perimeter of the triangle is

\[ s = r_1 + r_2 + \rho = r_1 + r_2 + \frac{r_1 r_2 (r_1 + r_2)}{r_1^3 + r_2 r_2 + r_2^2} = \frac{(r_1 + r_2)^3}{r_1^3 + r_2 r_2 + r_2^2} = \frac{(r_1 + r_2)^2 \rho}{r_1 r_2}. \]

The inradius of the triangle is the square root of

\[ \frac{r_1 r_2 \rho}{s} = \frac{r_1^2 r_2^2}{(r_1 + r_2)^2} = r^2. \]

It follows that this inradius is \( t \). The incircle of triangle O₁O₂O₃ is congruent to Archimedes’ twin circles. \( \Box \)
Construction. Let $M$ and $N$ be the midpoints of the semicircles $(AC)$ and $(CB)$ respectively. Construct

1. the lines $O_1N$ and $O_2M$ to intersect at $W_3$,
2. the circle with center $W_3$, passing through $C$ to intersect the semicircle $(AC)$ at $Q$ and $(CB)$ at $R$,
3. the lines $O_1Q$ and $O_2R$ to intersect at $O_3$.

The circle with center $O_3$ passing through $Q$ touches the semicircle $(CB)$ at $R$ and also the semicircle $(AB)$.
4.2.2 Alternative constructions of the incircle

**Theorem 4.4** (Bankoff). Let $P$ be the intersection (apart from $C$) of the circumcircles of the squares on $AC$ and $CB$. Let $Q$ be the intersection (apart from $C$) of the circumcircle of the square on $CB$ and the semicircle $(AC)$, and $R$ that of the circumcircle of the square on $AB$ and the semicircle $(CB)$. The points $P$, $Q$, $R$ are the points of tangency of the incircle of the arbelos with the semicircles.

**Proposition 4.5.** The intersection $S$ of the lines $AN$ and $BM$ also lies on the incircle of the arbelos, and the line $CS$ intersects $(AB)$ at $P$. 
Construction. Let $L'$ be the “lowest point” of the circle $(AB)$. Construct
(1) the line $L'C$ to intersect the semicircle $(AB)$ at $P$,
(2) the circle, center $L'$, through $A$ and $B$, to intersect the semicircles $(AC)$ and $(CB)$ at $Q$ and $R$.

Proposition 4.6. Let $X$ be the midpoint of the side of the square on $AC$ opposite to $AC$, and $Y$ that of the side of the square on $CB$ opposite to $CB$. The center $O_3$ of the incircle of the arbelos is the intersection of the lines $AY$ and $BX$. 
4.3 Archimedean circles

We shall call a circle Archimedean if it is congruent to Archimedes’ twin circle, i.e., with radius \( t = \frac{r_1r_2}{r_1 + r_2} \), and has further remarkable geometric properties.

1. (van Lamoen) The circle \((W_3)\) is tangent internally to the midway semicircle \((O_1O_2)\) at a point on the segment \(MN\). \(^1\)

2. (van Lamoem) The circle tangent to \(AB\) at \(O\) and to the midway semicircle is Archimedean. \(^2\)

3. (Schoch) Let \(MN\) intersect \(CD\) and \(OL\) at \(Q\) and \(K\) respectively. The smallest circles through \(Q\) and \(K\) tangent to the semicircle \(AB\) are Archimedean.

\(^1\)van Lamoen, June 10, 1999.
\(^2\)van Lamoen, June 10, 1999.
4. (a) The circle tangent to \((AB)\) and to the common tangent of \((AC)\) and \((CB)\) is Archimedean. 
   (b) The smallest circle through \(C\) tangent to \(AB\) is Archimedean.

5. Let \(EF\) be the common tangent of the semicircles \((AC)\) and \((CB)\). The smallest circles through \(E\) and \(F\) tangent to \(CD\) are Archimedean.

6. (Schoch) Let \(X\) and \(Y\) be the intersections of the semicircle \((AB)\) with the circles through \(C\), with centers \(A\) and \(B\) respectively. The smallest circle through \(X\) and \(Y\) tangent to \(CD\) are Archimedean.
7. (van Lamoen) Let $Y$ and $Z$ be the intersections of the midway semicircle with the semicircles $(AC)$ and $(CB)$. The circles with centers $Y$ and $Z$, each tangent to the line $CD$, are Archimedean.

8. (Schoch) (a) The circle tangent to the semicircle $(AB)$ and the circular arcs, with centers $A$ and $B$ respectively, each passing through $C$, is Archimedean.
(b) The circle with center on the Schoch line and tangent to both semicircles $(AC)$ and $(CB)$ is Archimedean.

9. (Woo) Let $\alpha$ be a positive real number. Consider the two circular arcs, each passing through $C$ and with centers $(-\alpha r_1, 0)$ and $(\alpha r_2, 0)$ respectively. The circle with center $U_\alpha$ on the Schoch line tangent to both of these arcs is Archimedean.

The Woo circle $(U_\alpha)$ which is tangent externally to the semicircle $(AB)$ touches it at $D$ (the intersection with the common tangent of $(AC)$ and $(CB)$).
10. (Power) Consider an arbelos with inner semicircles $C_1$ and $C_2$ of radii $a$ and $b$, and outer semicircle $C$ of radius $a + b$. It is known the Archimedean circles have radius $t = \frac{ab}{a+b}$. Let $Q_1$ and $Q_2$ be the “highest” points of $C_1$ and $C_2$ respectively.

A circle tangent to $(O)$ internally and to $OQ_1$ at $Q_1$ (or $OQ_2$ at $Q_2$) is Archimedean.

11. (van Lamoen)

12. (Bui)
5.1 Menelaus’ theorem

**Theorem 5.1** (Menelaus). Given a triangle $ABC$ with points $X$, $Y$, $Z$ on the side lines $BC$, $CA$, $AB$ respectively, the points $X$, $Y$, $Z$ are collinear if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$$ 

**Proof.** ($\implies$) Let $W$ be the point on $AC$ such that $BW//XY$. Then,

$$\frac{BX}{XC} = \frac{WY}{YC}, \quad \text{and} \quad \frac{AZ}{ZB} = \frac{AY}{YW}.$$ 

It follows that

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{WY}{YC} \cdot \frac{CY}{YA} \cdot \frac{AY}{YW} = -1.$$ 

($\impliedby$) Suppose the line joining $X$ and $Z$ intersects $AC$ at $Y'$. From above,

$$\frac{BX}{XC} \cdot \frac{CY'}{Y'A} \cdot \frac{AZ}{ZB} = -1 = \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB}.$$ 

It follows that

$$\frac{CY'}{Y'A} = \frac{CY}{YA}.$$ 

The points $Y'$ and $Y$ divide the segment $CA$ in the same ratio. These must be the same point, and $X$, $Y$, $Z$ are collinear. \qed
Example 5.1. The external angle bisectors of a triangle intersect their opposite sides at three collinear points.

Proof. If the external bisectors are $AX', BY', CZ'$ with $X', Y', Z'$ on $BC, CA, AB$ respectively, then

$$\frac{BX'}{X'C} = -\frac{c}{b}, \quad \frac{CY'}{Y'A} = -\frac{a}{c}, \quad \frac{AZ'}{Z'B} = -\frac{b}{a}.$$ 

It follows that $\frac{BX'}{X'C} \cdot \frac{CY'}{Y'A} \cdot \frac{AZ'}{Z'B} = -1$ and the points $X', Y', Z'$ are collinear. \qed
5.2 Centers of similitude of two circles

Given two circles $O(R)$ and $I(r)$, whose centers $O$ and $I$ are at a distance $d$ apart, we animate a point $X$ on $O(R)$ and construct a ray through $I$ oppositely parallel to the ray $OX$ to intersect the circle $I(r)$ at a point $Y$. The line joining $X$ and $Y$ intersects the line $OI$ of centers at a point $T$ which satisfies

$$OT : IT = OX : IY = R : r.$$  

This point $T$ is independent of the choice of $X$. It is called the internal center of similitude, or simply the insimilicenter, of the two circles.

If, on the other hand, we construct a ray through $I$ directly parallel to the ray $OX$ to intersect the circle $I(r)$ at $Y'$, the line $XY'$ always intersects $OI$ at another point $T'$. This is the external center of similitude, or simply the exsimilicenter, of the two circles. It divides the segment $OI$ in the ratio $OT' : T'I = R : -r$.

5.2.1 Desargue’s theorem

Given three circles with centers $A$, $B$, $C$ and distinct radii, show that the exsimilicenters of the three pairs of circles are collinear.
5.3 Ceva’s theorem

**Theorem 5.2 (Ceva).** Given a triangle $ABC$ with points $X, Y, Z$ on the side lines $BC$, $CA$, $AB$ respectively, the lines $AX, BY, CZ$ are concurrent if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = +1.$$

**Proof.** ($\implies$) Suppose the lines $AX, BY, CZ$ intersect at a point $P$. Consider the line $BPY$ cutting the sides of triangle $CAX$. By Menelaus’ theorem,

$$\frac{CY}{YA} \cdot \frac{AP}{PX} \cdot \frac{XB}{BC} = -1, \quad \text{or} \quad \frac{CY}{YA} \cdot \frac{PA}{XP} \cdot \frac{BX}{BC} = +1.$$

Also, consider the line $CPZ$ cutting the sides of triangle $ABX$. By Menelaus’ theorem again,

$$\frac{AZ}{ZB} \cdot \frac{BC}{CX} \cdot \frac{XP}{PA} = -1, \quad \text{or} \quad \frac{AZ}{ZB} \cdot \frac{BC}{XC} \cdot \frac{XP}{PA} = +1.$$

Multiplying the two equations together, we have

$$\frac{CY}{YA} \cdot \frac{AZ}{ZB} \cdot \frac{BX}{XC} = +1.$$

($\impliedby$) Exercise. \qed
5.4 Some triangle centers

5.4.1 The centroid

If $D$, $E$, $F$ are the midpoints of the sides $BC$, $CA$, $AB$ of triangle $ABC$, then clearly

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$ 

The medians $AD$, $BE$, $CF$ are therefore concurrent. Their intersection is the centroid $G$ of the triangle.

Consider triangle $ADC$ with transversal $BGE$. By Menelaus’ theorem,

$$-1 = \frac{AG}{GD} \cdot \frac{DB}{BC} \cdot \frac{CE}{EA} = \frac{AG}{GD} \cdot \frac{-1}{2} \cdot \frac{1}{1}.$$ 

It follows that $AG : GD = 2 : 1$. The centroid of a triangle divides each median in the ratio 2:1.

![Diagram of the centroid](image)

5.4.2 The incenter

Let $X$, $Y$, $Z$ be points on $BC$, $CA$, $AB$ such that $AX$, $BY$, $CZ$ bisect angles $BAC$, $CBA$ and $ACB$ respectively. Then

$$\frac{AZ}{ZB} = \frac{b}{a}, \quad \frac{BX}{XC} = \frac{c}{b}, \quad \frac{CY}{YA} = \frac{a}{c}.$$ 

It follows that

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{b}{a} \cdot \frac{c}{b} \cdot \frac{a}{c} = +1,$$

and $AX$, $BY$, $CZ$ are concurrent. Their intersection is the incenter of the triangle.

Applying Menelaus’ theorem to triangle $ABX$ with transversal $CIZ$, we have

$$-1 = \frac{AI}{IX} \cdot \frac{XC}{CB} \cdot \frac{BZ}{ZA} = \frac{AI}{IX} \cdot \frac{-b}{a} \cdot \frac{a}{b + c} \Rightarrow \frac{AI}{IX} = \frac{b + c}{a}.$$ 

![Diagram of the incenter](image)
5.4.3 The Gergonne point

Let the incircle of triangle $ABC$ be tangent to the sides $BC$ at $X$, $CA$ at $Y$, and $AB$ at $Z$ respectively. Since $AY = AZ = s - a$, $BZ = BX = s - b$, and $CX = CY = s - c$, we have

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{s - b}{s - c} \cdot \frac{s - c}{s - a} \cdot \frac{s - a}{s - b} = 1.$$ 

By Ceva’s theorem, the lines $AX$, $BY$, $CZ$ are concurrent. The intersection is called the Gergonne point $G_e$ of the triangle.

**Lemma 5.3.** The Gergonne point $G_e$ divides the cevian $AX$ in the ratio

$$\frac{AG_e}{G_eX} = \frac{a(s - a)}{(s - b)(s - c)}.$$ 

**Proof.** Applying Menelaus’ theorem to triangle $ABX$ with transversal $CG_eZ$, we have

$$-1 = \frac{AG_e}{G_eX} \cdot \frac{XC}{CB} \cdot \frac{BZ}{ZA} = \frac{AG_e}{G_eX} \cdot \frac{s - c}{a} \cdot \frac{s - b}{s - a} \quad \Rightarrow \quad \frac{AG_e}{G_eX} = \frac{a(s - a)}{(s - b)(s - c)}.$$ 

\[\blacksquare\]
### 5.4.4 The Nagel point

If \( X', Y', Z' \) are the points of tangency of the excircles with the respective sidelines, the lines \( AX', BY', CZ' \) are concurrent by Ceva’s theorem:

\[
\frac{BX'}{X'C} \cdot \frac{CY'}{Y'A} \cdot \frac{AZ'}{Z'B} = \frac{s-c}{s-b} \cdot \frac{s-c}{s-a} \cdot \frac{s-a}{s-b} = 1.
\]

The point of concurrency is the Nagel point \( N_a \).

**Lemma 5.4.** If the \( A \)-excircle of triangle \( ABC \) touches \( BC \) at \( X' \), then the Nagel point divides the cevian \( AX' \) in the ratio

\[
\frac{AN_a}{N_aX'} = \frac{a}{s-a}.
\]

**Proof.** Applying Menelaus’ theorem to triangle \( ACX' \) with transversal \( BN_aY' \), we have

\[
-1 = \frac{AN_a}{N_aX'} \cdot \frac{X'B}{BC} \cdot \frac{CY'}{Y'A} = \frac{AN_a}{N_aX'} \cdot \frac{- (s-c)}{a} \cdot \frac{s-a}{s-c} \implies \frac{AN_a}{N_aX'} = \frac{a}{s-a}.
\]
5.5 Isotomic conjugates

Given points $X$ on $BC$, $Y$ on $CA$, and $Z$ on $AB$, we consider their reflections in the midpoints of the respective sides. These are the points $X'$ on $BC$, $Y'$ on $CA$ and $Z'$ on $AB$ satisfying

$$BX' = XC, BX = X'C; \quad CY' = YA, CY = Y'A; \quad AZ' = ZB, AZ = Z'B.$$ 

Clearly, $AX, BY, CZ$ are concurrent if and only if $AX', BY', CZ'$ are concurrent.

![Diagram showing isotomic conjugates]

Proof.

$$\left(\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB}\right) \left(\frac{BX'}{XC'} \cdot \frac{CY'}{YA'} \cdot \frac{AZ'}{Z'B}\right) = \left(\frac{BX}{XC} \cdot \frac{BX'}{XC'}\right) \left(\frac{CY}{YA} \cdot \frac{CY'}{YA'}\right) \left(\frac{AZ}{ZB} \cdot \frac{AZ'}{Z'B}\right) = 1.$$ 

The points of concurrency of the two triads of lines are called isotomic conjugates.
Example 5.2. (The Gergonne and Nagel points)

Example 5.3. (The isotomic conjugate of the orthocenter) Let $H^*$ denote the isotomic conjugate of the orthocenter $H$. Its traces are the pedals of the reflection of $H$ in $O$. This latter point is the deLongchamps point $L_o$. 
Example 5.4. (Yff-Brocard points) Consider a point $P = (u : v : w)$ with traces $X, Y, Z$ satisfying $BX = CY = AZ = \mu$. This means that

$$\frac{w}{v + w} = \frac{u}{u + b} = \frac{v}{u + v} = \mu.$$

Elimination of $u, v, w$ leads to

$$0 = \begin{vmatrix} 0 & -\mu & a - \mu \\ b - \mu & 0 & -\mu \\ -\mu & c - \mu & 0 \end{vmatrix} = (a - \mu)(b - \mu)(c - \mu) - \mu^3.$$

Indeed, $\mu$ is the unique positive root of the cubic polynomial

$$(a - t)(b - t)(c - t) - t^3.$$

This gives the point

$$P = \left( \left( \frac{c - \mu}{b - \mu} \right)^{\frac{1}{3}} : \left( \frac{a - \mu}{c - \mu} \right)^{\frac{1}{3}} : \left( \frac{b - \mu}{a - \mu} \right)^{\frac{1}{3}} \right).$$

The isotomic conjugate

$$P^* = \left( \left( \frac{b - \mu}{c - \mu} \right)^{\frac{1}{3}} : \left( \frac{c - \mu}{a - \mu} \right)^{\frac{1}{3}} : \left( \frac{a - \mu}{b - \mu} \right)^{\frac{1}{3}} \right)$$

has traces $X', Y', Z'$ that satisfy

$$CX' = AY' = BZ' = \mu.$$

These points are called the Yff-Brocard points. \footnote{P. Yff, An analogue of the Brocard points, Amer. Math. Monthly, 70 (1963) 495 – 501.} They were briefly considered by A. L. Crelle. \footnote{A. L. Crelle, 1815.}
Chapter 6

The Euler line and the nine-point circle

6.1 The Euler line

6.1.1 Inferior and superior triangles

The inferior triangle of $ABC$ is the triangle $DEF$ whose vertices are the midpoints of the sides $BC$, $CA$, $AB$.

The two triangles share the same centroid $G$, and are homothetic at $G$ with ratio $-1:2$.

The superior triangle of $ABC$ is the triangle $A'B'C'$ bounded by the parallels of the sides through the opposite vertices.

The two triangles also share the same centroid $G$, and are homothetic at $G$ with ratio $2:-1$. 


6.1.2 The orthocenter and the Euler line

The three altitudes of a triangle are concurrent. This is because the line containing an altitude of triangle \(ABC\) is the perpendicular bisector of a side of its superior triangle. The three lines therefore intersect at the circumcenter of the superior triangle. This is the orthocenter of the given triangle.

The circumcenter, centroid, and orthocenter of a triangle are collinear. This is because the orthocenter, being the circumcenter of the superior triangle, is the image of the circumcenter under the homothety \(h(G, -2)\). The line containing them is called the Euler line of the reference triangle (provided it is non-equilateral).

The orthocenter of an acute (obtuse) triangle lies in the interior (exterior) of the triangle. The orthocenter of a right triangle is the right angle vertex.
6.2 The nine-point circle

**Theorem 6.1.** The following nine points associated with a triangle are on a circle whose center is the midpoint between the circumcenter and the orthocenter:

(i) the midpoints of the three sides,
(ii) the pedals (orthogonal projections) of the three vertices on their opposite sides,
(iii) the midpoints between the orthocenter and the three vertices.

\[ \text{Diagram} \]

**Proof.** (1) Let \( N \) be the circumcenter of the inferior triangle \( DEF \). Since \( DEF \) and \( ABC \) are homothetic at \( G \) in the ratio \( 1 : 2 \), \( N, G, O \) are collinear, and \( NG : GO = 1 : 2 \). Since \( HG : GO = 2 : 1 \), the four are collinear, and

\[ HN : NG : GO = 3 : 1 : 2, \]

and \( N \) is the midpoint of \( OH \).

(2) Let \( X \) be the pedal of \( H \) on \( BC \). Since \( N \) is the midpoint of \( OH \), the pedal of \( N \) is the midpoint of \( DX \). Therefore, \( N \) lies on the perpendicular bisector of \( DX \), and \( NX = ND \). Similarly, \( NE = NY \), and \( NF = NZ \) for the pedals of \( H \) on \( CA \) and \( AB \) respectively. This means that the circumcircle of \( DEF \) also contains \( X, Y, Z \).

(3) Let \( D', E', F' \) be the midpoints of \( AH, BH, CH \) respectively. The triangle \( D'E'F' \) is homothetic to \( ABC \) at \( H \) in the ratio \( 1 : 2 \). Denote by \( N' \) its circumcenter. The points \( N', G, O \) are collinear, and \( N'G : GO = 1 : 2 \). It follows that \( N' = N \), and the circumcircle of \( DEF \) also contains \( D', E', F' \). \( \square \)

This circle is called the **nine-point circle** of triangle \( ABC \). Its center \( N \) is called the nine-point center. Its radius is half of the circumradius of \( ABC \).
Theorem 6.2. Let $O_a$, $O_b$, $O_c$ be the reflections of the circumcenter $O$ in the sidelines $BC, CA, AB$ respectively.

1. The circle through $O_a$, $O_b$, $O_c$ is congruent to the circumcircle and has center at the orthocenter $H$.

2. The reflections of $H$ in the sidelines lie on the circumcircle.

Proof. (1) If $D$ is the midpoint of $BC$, $OO_a = 2OD = AH$. This means that $AHO_aO$ is a parallelogram, and $HO_a = AO$. Similarly, $HO_b = BO$ and $HO_c = CO$ for the other two reflections. Therefore, and $H$ is the center of the circle through $O_a$, $O_b$, $O_c$, and the circle is congruent to the circumcircle.

(2) If $H_a$ is the reflection of $H$ in $BC$, then $HH_aO_AO$ is a trapezoid symmetric in the line $BC$. Therefore, $OH_a = HO_a = OA$. This means that $H_a$ lies on the circumcircle; so do $H_b$ and $H_c$. □
6.3 Distances between triangle centers

6.3.1 Distance between the circumcenter and orthocenter

Proposition 6.3. $OH^2 = R^2(1 - 8 \cos \alpha \cos \beta \cos \gamma)$.

Proof. In triangle $AOH$, $AO = R$, $AH = 2R \cos \alpha$, and $\angle OAH = |\beta - \gamma|$. By the law of cosines,

\[
OH^2 = R^2(1 + 4 \cos^2 \alpha - 4 \cos \alpha \cos(\beta - \gamma))
= R^2(1 - 4 \cos \alpha (\cos(\beta + \gamma) + \cos(\beta - \gamma)))
= R^2(1 - 8 \cos \alpha \cos \beta \cos \gamma).
\]

\[\square\]
6.3.2 Distance between circumcenter and tritangent centers

Lemma 6.4. If the bisector of angle $A$ intersects the circumcircle at $M$, then $M$ is the center of the circle through $B$, $I$, $C$, and $I_a$.

Proof. (1) Since $M$ is the midpoint of the arc $BC$, $\angle MBC = \angle MCB = \angle MAB$. Therefore,

$$\angle MBI = \angle MBC + \angle CBI = \angle MAB + \angle IBA = \angle MIB,$$

and $MB = MI$. Similarly, $MC = MI$.

(2) On the other hand, since $\angle IBI_a$ and $\angle ICI_a$ are both right angles, the four points $B$, $I$, $C$, $I_aM$ are concyclic, with center at the midpoint of $II_A$. This is the point $M$. □

Theorem 6.5 (Euler). (a) $OI^2 = R^2 - 2Rr$.

(b) $OI_a^2 = R^2 + 2Rr_a$.

Proof. (a) Considering the power of $I$ in the circumcircle, we have

$$R^2 - OI^2 = AI \cdot IM = AI \cdot MB = \frac{r}{\sin \frac{\alpha}{2}} \cdot 2R \cdot \sin \frac{\alpha}{2} = 2Rr.$$

(b) Consider the power of $I_a$ in the circumcircle.

Note that $I_aA = \frac{r_a}{\sin \frac{\alpha}{2}}$. Also, $I_aM = MB = 2R \sin \frac{\alpha}{2}$.

$$OI_a^2 = R^2 + I_aA \cdot I_aM$$

$$= R^2 + \frac{r_a}{\sin \frac{\alpha}{2}} \cdot 2R \sin \frac{\alpha}{2}$$

$$= R^2 + 2Rr_a.$$
6.3 Distances between triangle centers

6.3.3 Distance between orthocenter and tritangent centers

Proposition 6.6.

\[ HI^2 = 2r^2 - 4R^2 \cos \alpha \cos \beta \cos \gamma, \]
\[ HI_a^2 = 2r_a^2 - 4R^2 \cos \alpha \cos \beta \cos \gamma. \]

Proof. In triangle \( AIH \), we have \( AH = 2R \cos \alpha \), \( AI = 4R \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \) and \( \angle HAI = \frac{|\beta - \gamma|}{2} \). By the law of cosines,

\[ HI^2 = AH^2 + AI^2 - 2AI \cdot AH \cdot \cos \frac{\beta - \gamma}{2} \]
\[ = 4R^2 \left( \cos^2 \alpha + 4 \sin^2 \frac{\beta}{2} \sin^2 \frac{\gamma}{2} - 4 \cos \alpha \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \cos \frac{\beta - \gamma}{2} \right) \]
\[ = 4R^2 \left( \cos^2 \alpha + 4 \sin^2 \frac{\beta}{2} \sin^2 \frac{\gamma}{2} - 4 \cos \alpha \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} - 4 \cos \alpha \sin^2 \frac{\beta}{2} \sin^2 \frac{\gamma}{2} \right) \]
\[ = 4R^2 \left( \cos^2 \alpha + 4 \sin^2 \frac{\beta}{2} \sin^2 \frac{\gamma}{2} - 4 \cos \alpha \sin \beta \sin \gamma - 4 \left( 1 - 2 \sin^2 \frac{\alpha}{2} \right) \sin^2 \frac{\beta}{2} \sin^2 \frac{\gamma}{2} \right) \]
\[ = 4R^2 \left( \cos \alpha (\cos \alpha - \sin \beta \sin \gamma) + 8 \sin^2 \frac{\alpha}{2} \sin^2 \frac{\beta}{2} \sin^2 \frac{\gamma}{2} \right) \]
\[ = 4R^2 \left( - \cos \alpha \cos \beta \cos \gamma + 8 \sin^2 \frac{\alpha}{2} \sin^2 \frac{\beta}{2} \sin^2 \frac{\gamma}{2} \right) \]
\[ = 2r^2 - 4R^2 \cos \alpha \cos \beta \cos \gamma. \]
(2) In triangle $\triangle AH_I$, $AI_a = 4R \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$.

By the law of cosines, we have

\[
HI_a^2 = AH^2 + AI_a^2 - 2AH \cdot AI_a \cdot \cos \frac{\beta - \gamma}{2}
\]

\[
= 4R^2 \left( \cos^2 \alpha + 4 \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2} - 4 \cos \alpha \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \cos \frac{\beta - \gamma}{2} \right)
\]

\[
= 4R^2 \left( \cos^2 \alpha + 4 \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2} - 4 \cos \alpha \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2} - 4 \cos \alpha \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \right)
\]

\[
= 4R^2 \left( \cos^2 \alpha + 4 \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2} - 4(1 - 2 \sin^2 \frac{\alpha}{2}) \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2} - \cos \alpha \sin \beta \sin \gamma \right)
\]

\[
= 4R^2 \left( \cos \alpha (\cos \alpha - \sin \beta \sin \gamma) + 8 \sin^2 \frac{\alpha}{2} \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2} \right)
\]

\[
= 4R^2 \left( - \cos \alpha \cos \beta \cos \gamma + 8 \sin^2 \frac{\alpha}{2} \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2} \right)
\]

\[
= 2r_a^2 - 4R^2 \cos \alpha \cos \beta \cos \gamma.
\]
6.4 Feuerbach’s theorem

Theorem 6.7 (Feuerbach). The nine-point circle is tangent internally to the incircle and externally to each of the excircles.

Proof. (1) Since $N$ is the midpoint of $OH$, $IN$ is a median of triangle $IOH$. By Apollonius’ theorem,

$$NI^2 = \frac{1}{2}(IH^2 + OI^2) - \frac{1}{4}OH^2$$

$$= \frac{1}{4}R^2 - Rr + r^2$$

$$= \left(\frac{R}{2} - r\right)^2.$$

Therefore, $NI$ is the difference between the radii of the nine-point circle and the incircle. This shows that the two circles are tangent to each other internally.

(2) Similarly, in triangle $I_aOH$,

$$NI_a^2 = \frac{1}{2}(HI_a^2 + OI_a^2) - \frac{1}{4}OH^2$$

$$= \frac{1}{4}R^2 + Rr_a + r_a^2$$

$$= \left(\frac{R}{2} + r_a\right)^2.$$

This shows that the distance between the centers of the nine-point and an excircle is the sum of their radii. The two circles are tangent externally.

The point of tangency $F_e$ of the incircle and the nine-point circle is called the Feuerbach point.
The Euler line and the nine-point circle

[Diagram showing geometric relationships and points A, B, C, Ia, Ib, Ic, N, F, Fa, Fb, Fe, Fc, Aa, Ab, Ac, Ba, Bc, Ca]
Chapter 7

Isogonal conjugates

7.1 Directed angles

A reference triangle $ABC$ in a plane induces an orientation of the plane, with respect to which all angles are signed. For two given lines $L$ and $L'$, the directed angle $\angle(L, L')$ between them is the angle of rotation from $L$ to $L'$ in the induced orientation of the plane. It takes values of modulo $\pi$. The following basic properties of directed angles make many geometric reasoning simple without the reference of a diagram.

**Theorem 7.1.**

1. $\angle(L', L) = -\angle(L, L').$
2. $\angle(L_1, L_2) + \angle(L_2, L_3) = \angle(L_1, L_3)$ for any three lines $L_1$, $L_2$ and $L_3$.
3. Four points $P$, $Q$, $X$, $Y$ are concyclic if and only if $\angle(PX, XQ) = \angle(PY, YQ)$.

*Remark.* In calculations with directed angles, we shall slightly abuse notations by using the equality sign instead of the sign for congruence modulo $\pi$. It is understood that directed angles are defined up to multiples of $\pi$. For example, we shall write $\beta + \gamma = -\alpha$ even though it should be more properly $\beta + \gamma = \pi - \alpha$ or $\beta + \gamma \equiv -\alpha \mod \pi$.

*Exercise*

1. If $a$, $b$, $c$ are the sidelines of triangle $ABC$, then $\angle(a, b) = -\gamma$ etc.
7.2 Isogonal conjugates

Let $P$ be a given point. Consider the reflections of the cevians $AP$, $BP$, $CP$ in the respective bisectors of angles $A$, $B$, $C$, i.e., By Ceva’s theorem, these reflections are concurrent. Their intersection is the isogonal conjugate of $P$.

Let $P$ and $Q$ be isogonal conjugates, $AP$ and $AQ$ intersecting $BC$ at $X$ and $X'$ respectively. Then

$$\frac{BX}{XC} \cdot \frac{BX'}{X'C} = \frac{c^2}{b^2}.$$

**Example 7.1.** The incenter is the isogonal conjugate of itself. The same is true for the excenters.

**Example 7.2.** (The circumcenter and orthocenter) For a given triangle with circumcenter $O$, the line $OA$ and the altitude through $A$ are isogonal lines, similarly for the circumradii and altitudes through $B$ and $C$. Since the circumradii are concurrent at $O$, the altitudes also are concurrent. Their intersection is the orthocenter $H$, which is the isogonal conjugate of $O$. 
7.3 The symmedian point and the centroid

The isogonal lines of the medians are called the **symmedians**. The isogonal conjugate of the centroid $G$ is called the **symmedian point** $K$ of the triangle.

Consider triangle $ABC$ together with its *tangential triangle* $A'B'C'$, the triangle bounded by the tangents of the circumcircle at the vertices.

Since $A'$ is equidistant from $B$ and $C$, we construct the circle $A'(B) = A'(C)$ and extend the sides $AB$ and $AC$ to meet this circle again at $Z$ and $Y$ respectively. Note that

$$\angle(A'Y, A'B') = \pi - 2(\pi - \alpha - \gamma) = \pi - 2\beta,$$
and similarly, $\angle(A'C', A'Z') = \pi - 2\gamma$. Since $\angle(A'B', A'C') = \pi - 2\alpha$, we have

$$\angle(AY, A'Z) = \angle(AY, A'B') + \angle(A'B', A'C') + \angle(A'C', A'Z)$$

$$= (\pi - 2\beta) + (\pi - 2\alpha) + (\pi - 2\gamma)$$

$$= \pi$$

$$\equiv 0 \mod \pi.$$ This shows that $Y, A'$ and $Z'$ are collinear, so that

(i) $AA'$ is a median of triangle $AYZ$,

(ii) $AYZ$ and $ABC$ are similar.

It follows that $AA'$ is the isogonal line of the $A$-median, i.e., a symmedian. Similarly, the $BB'$ and $CC'$ are the symmedians isogonal to $B$- and $C$-medians. The lines $AA', BB', CC'$ therefore intersect at the isogonal conjugate of the centroid $G$. 
7.4 Isogonal conjugates of the Gergonne and Nagel points

7.4.1 The Gergonne point and the insimilicenter $T_+$

Consider the intouch triangle $DEF$ of triangle $ABC$.

(1) If $D'$ is the reflection of $D$ in the bisector $AI$, then
(i) $D'$ is a point on the incircle, and
(ii) the lines $AD$ and $AD'$ are isogonal with respect to $A$.

(2) Likewise, $E'$ and $F'$ are the reflections of $E$ and $F$ in the bisectors $BI$ and $CI$ respectively, then
(i) these are points on the incircle,
(ii) the lines $BE'$ and $CF'$ are isogonals of $BE$ and $CF$ with respect to angles $B$ and $C$.

Therefore, the lines $AD'$, $BE'$, and $CF'$ concur at the isogonal conjugate of the Gergonne point.

(3) In fact, $E'F'$ is parallel to $BC$.

This follows from
\[(ID, IE') = (ID, IE) + (IE, IE')\]
\[= (ID, IE) + 2(IE, IB)\]
\[= (ID, IE) + 2((IE, AC) + (AC, IB))\]
\[= (ID, IE) + 2(AC, IB) \quad \text{since} \ (IE, AC) = \frac{\pi}{2}\]
\[= (\pi - \gamma) + 2 \left( \gamma + \frac{\beta}{2} \right)\]
\[= \beta + \gamma = -\alpha \quad \text{(mod \(\pi\));}\]

\[(ID, IF') = (ID, IF) + (IF, IF')\]
\[= (ID, IF) + 2(IF, IC)\]
\[= (ID, IF) + 2((IF, AB) + (AB, IC))\]
\[= (ID, IF) + 2(AB, IC) \quad \text{since} \ (IF, AB) = \frac{\pi}{2}\]
\[= - (\pi - \beta) - 2 \left( \beta + \frac{\gamma}{2} \right)\]
\[= - (\beta + \gamma) = \alpha \quad \text{(mod \(\pi\))}\]

Since \(E'\) and \(F'\) are on the incircle, and \(ID \perp BC\), it follows that \(E'F'\) is parallel to \(BC\).

(4) Similarly, \(F'D'\) and \(D'E'\) are parallel to \(CA\) and \(AB\) respectively. It follows that \(D'E'F'\) is homothetic to \(ABC\).

The ratio of homothety is \(r : R\). Therefore, the center of homothety is the point \(T_+\) which divides \(OI\) in the ratio \(R : r\). This is the internal center of similitude, or simply the insimilicenter of \((O)\) and \((I)\).
7.4 Isogonal conjugates of the Gergonne and Nagel points

7.4.2 The Nagel point and the exsimilicenter $T_-$

The isogonal conjugate of the Nagel point is the point $T_-$ which divides $OI$ in the ratio $OT_- : T_- I = R : -r$. This is the external center of similitude (or exsimilicenter) of the circumcircle and the incircle.
7.5 The Brocard points

Analogous to the Crelles points, we may ask if there are concurrent lines through the vertices making equal angles with the sidelines. More precisely, given triangle $ABC$, does there exist a point $P$ satisfying

$$\angle BAP = \angle CBP = \angle ACP = \omega.$$ 

It turns out that is one such unique configuration.

Note that if $P$ is a point satisfying $\angle BAP = \angle CBP$, then the circle through $P, A, B$ is tangent to $BC$ at $B$. This circle is unique and can be constructed as follows. Its center is the intersection of the perpendicular bisector of $AB$ and the perpendicular to $BC$ at $B$.

Likewise, if $\angle CBP = \angle ACP$, then the circle through $P, B, C$ is tangent to $CA$ at $C$. It follows that $P$ is the intersection of these two circles. With this $P$, the circle $PCA$ is tangent to $AB$ at $A$.

By Ceva’s theorem, the angle $\omega$ satisfies the equation

$$\sin^3 \omega = \sin(\beta - \omega) \sin(\alpha - \omega) \sin(\gamma - \omega).$$

It also follows that with the same $\omega$, there is another triad of circles intersecting at another point $Q$ such that

$$\angle CAQ = \angle ABQ = \angle BCQ = \omega.$$
The points $P$ and $Q$ are isogonal conjugates. They are called the Brocard points of triangle $ABC$. 
7.6 Kariya’s theorem

Given a triangle $ABC$ with incenter $I$, consider a point $X$ on the perpendicular from $I$ to $BC$, such that $IX = t$. We regard $t > 0$ if $X$ and the point of tangency of the incircle with the side $BC$ are on the same side of $I$.

**Theorem 7.2 (Kariya).** Let $I$ be the incenter of triangle $ABC$. If points $X$, $Y$, $Z$ are chosen on the perpendiculars from $I$ to $BC$, $CA$, $AB$ respectively such that $IX = IY = IZ$, then the lines $AX$, $BY$, $CZ$ are concurrent.

![Diagram of triangle ABC with incenter I and points X, Y, Z on perpendiculars]

**Proof.** (1) We compute the length of $AX$. Let the perpendicular from $A$ to $BC$ and the parallel from $X$ to the same line intersect at $X'$. In the right triangle $AXX'$,

\[
AX' = \frac{2\Delta}{a} - r + t = \frac{2rs}{a} - r + t = \frac{r(b + c)}{a} + t,
\]

\[
XX' = (s - b) - c \cos B
\]

\[
= \frac{1}{2}(c + a - b) - \frac{1}{2a}(c^2 + a^2 - b^2)
\]

\[
= \frac{2a}{a}(c + a - b) - (c^2 + a^2 - b^2)
\]

\[
= \frac{b^2 - c^2 - a(b - c)}{2a} = \frac{(b - c)(b + c - a)}{2a} = \frac{(b - c)(s - a)}{a}.
\]

Applying the Pythagorean theorem to the right triangle $AXX'$, we have
(2) Let \( M' \) be the midpoint of the arc \( BAC \) of the circumcircle, and let \( MX \) intersect \( OI \) at \( P \). We shall prove that angle \( IAP = \angle IAX \).

First of all,

\[
AI^2 = \frac{(s-a)^2}{\cos^2 A} = (s-a)^2 \cdot \frac{bc}{s(s-a)} = \frac{bc(s-a)}{s} = \frac{4Rr(s-a)}{a}.
\]

For later use, we also establish

\[
AI^2 + 2Rr = \frac{4Rr(s-a)}{a} + 2Rr = \frac{2Rr(2s-2a) + 2Rr \cdot a}{a} = \frac{2Rr(b+c)}{a}.
\]

Since \( IP : PO = t : R, IP = \frac{t}{R+r} \cdot OI \). Recall that \( OI^2 = R^2 - 2Rr \). Applying the law
of cosines to triangle $AIP$, we have

$$AP^2 = AI^2 + IP^2 - 2 \cdot AI \cdot IP \cos AIP$$

$$= AI^2 + \left(\frac{t^2}{(R + t)^2}\right) (R^2 - 2Rr) - \frac{t}{R + t} (AI^2 + OI^2 - R^2)$$

$$= AI^2 + \left(\frac{t^2}{(R + t)^2}\right) (R^2 - 2Rr) - \frac{t}{R + t} (AI^2 - 2Rr)$$

$$= \frac{R}{R + t} \cdot AI^2 + \frac{2R^2rt}{(R + t)^2} + \frac{R^2t^2}{(R + t)^2}$$

$$= \frac{R(R + t)AI^2 + 2R^2rt + R^2t^2}{(R + t)^2}$$

$$= \frac{R^2 \cdot AI^2 + 2Rr(4R^2t + R^2t^2)}{a(R + t)^2}$$

$$= \frac{R^2(4Rr(s - a) + 2r(b + c)t + at^2)}{a(R + t)^2}.$$

Note that $AP^2 = \frac{R^2}{(R + t)^2} \cdot AX^2$. This means that $AP = \frac{R}{R + t} \cdot AX$.

(3) Let $AI$ intersect $PX$ at $X''$ and the circumcircle again at $M$, the antipode of $M'$. Note that $\frac{MM'}{MO} = -2$ and $\frac{OI}{IP} = -\frac{R + t}{t}$.

Applying Menelaus’ theorem to triangle $M'OP$ and transversal $IX''M$, we have

$$-1 = \frac{PX''}{X''M'} \cdot \frac{M'M}{MO} \cdot \frac{OI}{IP} \Rightarrow \frac{PX''}{X''M'} = -\frac{t}{2(R + t)} \Rightarrow \frac{PX''}{PM'} = -\frac{t}{2R + t}.$$

Now $\frac{XP}{PM'} = \frac{t}{R}$. Therefore, $\frac{XP}{PM'} = \frac{2R + t}{-R}$, and

$$\frac{XX''}{X''P} = \frac{R + t}{R} = \frac{AX}{AP}.$$
This shows that $AX''$ bisects angle $XAP$. Since $AX''$ is the bisector of angle $A$, the lines $AP$ and $AX$ are isogonal with respect to angle $A$.

(4) Likewise, if points $Y$ and $Z$ are chosen on the perpendiculars from $I$ to $CA$ and $AB$ such that $IY = IZ = t = IX$, then with the same point $P$ on $OI$, the lines $BP$ and $BY$ are isogonal with respect to angle $B$, and $CP$, $CZ$ isogonal with respect to angle $C$. Therefore the three lines $AX$, $BY$, $CZ$ intersect at the isogonal conjugate of $P$ (which divides $OI$ in the ratio $OP : PI = R : t$).

7.7 Isogonal conjugate of an infinite point

**Proposition 7.3.** Given a triangle $ABC$ and a line $\ell$, let $\ell_a$, $\ell_b$, $\ell_c$ be the parallels to $\ell$ through $A$, $B$, $C$ respectively, and $\ell'_a$, $\ell'_b$, $\ell'_c$ their reflections in the angle bisectors $AI$, $BI$, $CI$ respectively. The lines $\ell'_a$, $\ell'_b$, $\ell'_c$ intersect at a point on the circumcircle of triangle $ABC$.

![Diagram of isogonal conjugate](image)

**Proof.** Let $P$ be the intersection of $\ell'_b$ and $\ell'_c$.

\[
(BP, PC) = (\ell'_b, \ell'_c)
= (\ell'_b, IB) + (IB, IC) + (IC, \ell'_c)
= (IB, \ell_b) + (IB, IC) + (\ell_c, IC)
= (IB, \ell) + (IB, IC) + (\ell, IC)
= 2(IB, IC)
= 2\left(\frac{\pi}{2} + \frac{A}{2}\right)
= (BA, AC) \quad \text{(mod } \pi).\]

Therefore, $\ell'_b$ and $\ell'_c$ intersect at a point on the circumcircle of triangle $ABC$.

Similarly, $\ell'_a$ and $\ell'_b$ intersect at a point $P'$ on the circumcircle. Clearly, $P$ and $P'$ are the same point since they are both on the reflection of $\ell_b$ in the bisector $IB$. Therefore, the three reflections $\ell'_a$, $\ell'_b$, and $\ell'_c$ intersect at the same point on the circumcircle.
**Proposition 7.4.** The isogonal conjugates of the infinite points of two perpendicular lines are antipodal points on the circumcircle.

**Proof.** If $P$ and $Q$ are the isogonal conjugates of the infinite points of two perpendicular lines $\ell$ and $\ell'$ through $A$, then $AP$ and $AQ$ are the reflections of $\ell$ and $\ell'$ in the bisector $AI$.

\[(AP, AQ) = (AP, IA) + (IA, AQ) = -(\ell, IA) - (IA, \ell') = -(\ell, \ell') = \frac{\pi}{2}.\]

Therefore, $P$ and $Q$ are antipodal points. \qed
Chapter 8

The excentral triangle

8.1 The Euler line of the excentral triangle

The *excentral triangle* has vertices the excenters $I_a$, $I_b$, $I_c$. Its sides are the external bisectors of the angles of triangle $ABC$, so that the internal bisectors are its altitudes, and $I$ its orthocenter. Thus, $ABC$ is the orthic triangle of $I_aI_bI_c$, and its circumcircle is the nine-point circle of the excentral triangle. Therefore, the line $OI$ is the Euler line of the excentral triangle.

<table>
<thead>
<tr>
<th>Triangle center</th>
<th>Point on $OI$</th>
<th>Division ratio of $OI$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orthocenter</td>
<td>$I$</td>
<td>$1 : 0$</td>
</tr>
<tr>
<td>Nine-point center</td>
<td>$O$</td>
<td>$0 : 1$</td>
</tr>
<tr>
<td>Circumcenter</td>
<td>Reflection of $I$ in $O$</td>
<td>$-1 : 2$</td>
</tr>
<tr>
<td>Centroid</td>
<td></td>
<td>$-1 : 4$</td>
</tr>
</tbody>
</table>

Exercise

1. The point $E_t$ of the excentral triangle divides $OI$ in the ratio $-1 + 2t : 2(1 - t)$. 
8.2 The circumcircle of the excentral triangle

We also have the following interesting facts.

(i) The midpoints of the segments $II_a, II_b,$ and $II_c$ are on the circumcircle of $ABC$.
(ii) The midpoints of $I_bI_c, I_cI_a,$ and $I_aI_b$ are also on the circumcircle of $ABC$.
(iii) In particular, the midpoints $M \text{ of } II_a$ and $M' \text{ of } I_bI_c$ are antipodal on the circumcircle, and $MM'$ is the perpendicular bisector of $BC$.

More surprising are the following facts about the circumcircle of the excentral triangle.

(iv) The circumradius of the excentral circle is $2R$.
(v) Since the excentral triangle has nine-point center $O$ and orthocenter $I$, its circumcenter is the reflection of $I$ in $O$, i.e., $I' := 2O - I$.

Note that the line $I'I_a$ is perpendicular to $BC$. It therefore contains the point of tangency with the $A$-excircle. From this, we deduce two more interesting facts.
**Proposition 8.1.** The perpendiculars from three excenters to the corresponding sides concur at the reflection of the incenter in the circumcenter.
Theorem 8.2. \( r_a + r_b + r_c = 4R + r. \)

**Proof.** Consider the diameter \( MM' \) of the circumcircle (of \( ABC \)) perpendicular to \( BC \).

(i) \( r_b + r_c = 2 \cdot DM' = 2(R + OD). \)

(ii) \( 2 \cdot OD = r + I'A_a = r + 2R - r_a. \)

(iii) It follows that \( r_a + r_b + r_c = 4R + r. \) \( \square \)
Chapter 9

Homogeneous Barycentric Coordinates

9.1 Absolute and homogeneous barycentric coordinates

The notion of barycentric coordinates dates back to A. F. Möbius (–). Given a reference triangle $ABC$, we put at the vertices $A$, $B$, $C$ masses $u$, $v$, $w$ respectively, and determine the balance point. The masses at $B$ and $C$ can be replaced by a single mass $v + w$ at the point $X = \frac{vB+wC}{v+w}$. Together with the mass at $A$, this can be replaced by a mass $u + v + w$ at the point $P$ which divides $AX$ in the ratio $AP : PX = v + w : u$. This is the point with absolute barycentric coordinate $\frac{uA+vB+wC}{u+v+w}$, provided $u + v + w \neq 0$. We also say that the balance point $P$ has homogeneous barycentric coordinates $(u : v : w)$ with reference to $ABC$.

9.1.1 The centroid

The midpoints of the sides are

$$D = \frac{B + C}{2}, \quad E = \frac{C + A}{2}, \quad F = \frac{A + B}{2}.$$ 

The centroid $G$ divides each median in the ratio 2 : 1. Thus,

$$G = \frac{A + 2D}{3} = \frac{A + B + C}{3}.$$ 

This is the absolute barycentric coordinate of $G$ (with reference to $ABC$). Its homogeneous barycentric coordinates are simply

$$G = (1 : 1 : 1).$$

---

1 A triple $(u : v : w)$ with $u + v + w = 0$ does not represent any finite point on the plane. We shall say that it represents an infinite point. See §?.

9.1.2 The incenter

The bisector $AX$ divides the side $BC$ in the ratio $BX : XC = c : b$. This gives $X = \frac{bB+cC}{b+c}$ · Note that $BX$ has length $\frac{ca}{b+c}$. Now, in triangle $ABX$, the bisector $BI$ divides $AX$ in the ratio $AI : IX = c : \frac{ca}{b+c} = b+c : a$. It follows that

$$I = \frac{aA + (b+c)X}{a+b+c} = \frac{aA + bB + cC}{a+b+c}.$$ 

The homogeneous barycentric coordinates of the incenter are

$$I = (a : b : c).$$

9.1.3 The barycenter of the perimeter

Consider the barycenter (center of mass) of the perimeter of triangle $ABC$. The edges $BC, CA, AB$ can be replaced respectively by masses $a, b, c$ at their midpoint $D = \frac{B+C}{2}$, $E = \frac{C+A}{2}$, and $F = \frac{A+B}{2}$. With reference to the medial triangle $DEF$, this has coordinates $a : b : c$. Since the sidelengths of triangle $DEF$ are in the same proportions, this barycenter is the incenter of the medial triangle, also called the Spieker center $S_p$ of $ABC$. 
The center of mass of the perimeter is therefore the point
\[ S_p = \frac{a \cdot D + b \cdot E + c \cdot F}{a + b + c} \]
\[ = \frac{a \cdot \frac{B+C}{2} + b \cdot \frac{C+A}{2} + c \cdot \frac{A+B}{2}}{a + b + c} \]
\[ = \frac{(b + c)A + (c + a)B + (a + b)C}{2(a + b + c)}. \]

In homogeneous barycentric coordinates,
\[ S_p = (b + c : c + a : a + b). \]

### 9.1.4 The Gergonne point

We follow the same method to compute the coordinates of the Gergonne point \( G_e \). Here, \( BX = s - b \) and \( XC = s - c \), so that
\[ X = \frac{(s - b)b + (s - c)C}{a}. \]

The ratio \( AG_e : G_e X \), however, is not immediate obvious. It can nevertheless be found by applying the Menelaus theorem to triangle \( ABX \) with transversal \( CZ \). Thus,
\[ \frac{AG_e}{G_e X} \cdot \frac{XC}{CB} \cdot \frac{BZ}{ZA} = -1. \]

From this,
\[ \frac{AG_e}{G_e X} = -\frac{CB}{XC} \cdot \frac{ZA}{BZ} = -\frac{-a}{s - c} \cdot \frac{s - a}{s - b} = \frac{a(s - a)}{(s - b)(s - c)}. \]

Therefore,
\[ G_e = \frac{(s - b)(s - c)A + a(s - a)X}{(s - b)(s - c) + a(s - a)} \]
\[ = \frac{(s - b)(s - c)A + (s - a)(s - c)B + (s - a)(s - b)C}{(s - b)(s - c) + a(s - a)}. \]
The homogeneous barycentric coordinates of the Gergonne point are

\[
G_e = \left( s - b \right) \left( s - c \right) : \left( s - c \right) \left( s - a \right) : \left( s - a \right) \left( s - b \right) \\
= \frac{1}{s - a} : \frac{1}{s - b} : \frac{1}{s - c}.
\]

### 9.2 Cevian triangle

It is clear that the calculations in the preceding section applies in the general case. We summarize the results in the following useful alternative of the Ceva theorem.

**Theorem 9.1 (Ceva).** Let \( X, Y, Z \) be points on the lines \( BC, CA, AB \) respectively. The lines \( AX, BY, CZ \) are collinear if and only if the given points have coordinates of the form

\[
X = (0 : y : z), \\
Y = (x : 0 : z), \\
Z = (x : y : 0),
\]

for some \( x, y, z \). If this condition is satisfied, the common point of the lines \( AX, BY, CZ \) is \( P = (x : y : z) \).

**Remarks.** (1) The points \( X, Y, Z \) are called the traces of \( P \). We also say that \( XYZ \) is the cevian triangle of \( P \) (with reference to triangle \( ABC \)). Sometimes, we shall adopt the more functional notation for the cevian triangle and its vertices:

\[
\text{cev}(P) : \quad \begin{align*}
A_P &= (0 : y : z), \\
B_P &= (x : 0 : z), \\
C_P &= (x : y : 0).
\end{align*}
\]

(2) The point \( P \) divides the segment \( AX \) in the ratio \( PX : AX = x : x + y + z \).

(3) It follows that the areas of the oriented triangles \( PBC \) and \( ABC \) are in the ratio \( \Delta(PBC) : \Delta(ABC) = x : x + y + z \). This leads to the following interpretation of homogeneous barycentric coordinates: the homogeneous barycentric coordinates of a point \( P \) can be taken as the proportions of (signed) areas of oriented triangles:

\[
P = \Delta(PBC) : \Delta(PCA) : \Delta(PAB).
\]
9.2 Cevian triangle

9.2.1 The Nagel point and the extouch triangle

The $A$-excircle touches the side $BC$ at a point $X'$ such that $BX' = s - c$ and $X'C = s - b$. From this, the homogeneous barycentric coordinates of $X'$ are $0 : s - b : s - c$; similarly for the points of tangency $Y'$ and $Z$ of the $B$- and $C$-excircles:

$$X' = (0 : s - b : s - c),$$
$$Y' = (s - a : 0 : s - c),$$
$$Z' = (s - a : s - b : 0),$$

From these we conclude that $AX'$, $BY'$, and $CZ'$ concur. Their common point is called the Nagel point and has coordinates

$$N_a = (s - a : s - b : s - c).$$

The triangle $X'Y'Z'$ is called the extouch triangle.

9.2.2 The orthocenter and the orthic triangle

For the orthocenter $H$ with traces $X$, $Y$, $Z$ on $BC$, $CA$, $AB$ respectively, we have $BX = c \cos B$, $XC = b \cos C$. This gives

$$BX : XC = c \cos B : b \cos C = \frac{\cos B}{b} : \frac{\cos C}{c};$$

similarly for the other two traces.
The triangle $XYZ$ is called the orthic triangle.

**Exercise**

1. In triangle $ABC$, $Z$ is on $AB$ such that $AZ : ZB = 1 : 2$ and $Y$ is on $AC$ such that $AY : YC = 4 : 3$. Let $P$ be the intersection of $BY$ and $CZ$, and let $X$ be the intersection of $BC$ and ray $AP$. Find the coordinates of $P$ and the ratio $AP : PX$.

2. Find the area of the anticevian triangle of $(u : v : w)$.

3. The anticevian triangles of $O$ and $K$ have equal areas.
4. The areas of the orthic triangle, the reference triangle, and the anticevian triangle of $O$ are in geometric progression.

5. Find the coordinates of $P = (u : v : w)$ in its anticevian triangle. \(^2\)

6. If $P$ does not lie on the sidelines of triangle $ABC$ and is the centroid of its own anticevian triangle, show that $P$ is the centroid of triangle $ABC$.

### 9.2.3 The inferior and superior triangles

The triangle whose vertices are the midpoints of the sides of triangle $ABC$ is called the inferior triangle of $ABC$. Its vertices are the points $A' = (0 : 1 : 1)$, $B' = (1 : 0 : 1)$, $C' = (1 : 1 : 0)$.

This triangle is homothetic to $ABC$ at the centroid $G$, with ratio $-\frac{1}{2}$. Equivalently, we say that it is the image of $ABC$ under the homothety $h(G, -\frac{1}{2})$. This means that $h(G, -\frac{1}{2})$ is a one-to-one correspondence of points on the plane such that $P$ and $P'$ have the same homogeneous barycentric coordinates with reference to $ABC$ and $A'B'C'$ whenever $X, X'$ and $G$ are collinear and

$$\frac{GP'}{GP} = -\frac{1}{2}.$$

We call $P'$ the inferior of $P$. More explicitly, $P'$ divides $PG$ in the ratio $PP' : P'G = 3 : -1$, so that $P' = \frac{1}{2}(3G - P)$. Suppose $P$ has homogeneous barycentric coordinates $(x : y : z)$ with reference to $ABC$. It is the point $P = \frac{xA + yB + zC}{x + y + z}$. \(^3\) Thus,

$$P' = \frac{1}{2}(3G - P)$$

$$= \frac{1}{2} \left( A + B + C - \frac{xA + yB + zC}{x + y + z} \right)$$

$$= \frac{1}{2} \left( \frac{(x + y + z)(A + B + C) - (xA + yB + zC)}{x + y + z} \right)$$

$$= \frac{(y + z)A + (z + x)B + (x + y)C}{2(x + y + z)}.$$

It follows that the homogenous coordinates of $P'$ are $(y + z : z + x : x + y)$.

The superior triangle of $ABC$ is its image under the homothety $h(G, -2)$. Its vertices are the points $A'' = (-1 : 1 : 1)$, $B'' = (1 : -1 : 1)$, $C'' = (1 : 1 : -1)$.

\(^2\)\((v + w - u : w + u - v : u + v - w)\).

\(^3\)The conversion from homogeneous barycentric coordinates to absolute barycentric coordinates is called normalization.
Chapter 10

Some applications of barycentric coordinates

10.1 Construction of mixtilinear incircles

10.1.1 The insimilicenter and the exsimilicenter of the circumcircle and incircle

The centers of similitude of two circles are the points dividing the centers in the ratio of their radii, either internally or externally. For the circumcircle and the incircle, these are

\[ T_+ = \frac{1}{R + r} (r \cdot O + R \cdot I), \]
\[ T_- = \frac{1}{R - r} (-r \cdot O + R \cdot I). \]

We give an interesting application of these centers of similitude.
10.1.2 Mixtilinear incircles

A mixtilinear incircle of triangle $ABC$ is one that is tangent to two sides of the triangle and to the circumcircle internally. Denote by $A'$ the point of tangency of the mixtilinear incircle $K(\rho)$ in angle $A$ with the circumcircle. The center $K$ clearly lies on the bisector of angle $A$, and $AK : KI = \rho : -(\rho - r)$. In terms of barycentric coordinates,

$$K = \frac{1}{r} \left( -(\rho - r)A + \rho I \right).$$

Also, since the circumcircle $O(A')$ and the mixtilinear incircle $K(A')$ touch each other at $A'$, we have $OK : KA' = R - \rho : \rho$, where $R$ is the circumradius. From this,

$$K = \frac{1}{R} \left( \rho O + (R - \rho)A' \right).$$

Comparing these two equations, we obtain, by rearranging terms,

$$\frac{RI - rO}{R - r} = \frac{R(\rho - r)A + r(R - \rho)A'}{\rho(R - r)}.$$

We note some interesting consequences of this formula. First of all, it gives the intersection of the lines joining $AA'$ and $OI$. Note that the point on the line $OI$ represented by the left hand side is $T_-$, the exsimilicenter of the circumcircle and the incircle.

This leads to a simple construction of the mixtilinear incircle. Given a triangle $ABC$, extend $AT_-$ to intersect the circumcircle at $A'$. The intersection of $AI$ and $A'O$ is the center $K_A$ of the mixtilinear incircle in angle $A$.

The other two mixtilinear incircles can be constructed similarly.
The Gergonne and Nagel points are examples of isotomic conjugates. Two points \( P \) and \( Q \) (not on any of the side lines of the reference triangle) are said to be isotomic conjugates if their respective traces are symmetric with respect to the midpoints of the corresponding sides. Thus,

\[
BX = X'C, \quad CY = Y'A, \quad AZ = Z'B.
\]

We shall denote the \textit{isotomic conjugate} of \( P \) by \( P^\bullet \). If \( P = (x : y : z) \), then

\[
P^\bullet = \left( \frac{1}{x} : \frac{1}{y} : \frac{1}{z} \right) = (yz : zx : xy).
\]
**Example 10.1.** (The Gergonne and Nagel points)

\[ G_e = \left( \frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c} \right), \quad N_a = (s - a : s - b : s - c). \]

**Example 10.2.** (The isotomic conjugate of the orthocenter) The isotomic conjugate of the orthocenter is the point

\[ H^\bullet = (b^2 + c^2 - a^2 : c^2 + a^2 - b^2 : a^2 + b^2 - c^2). \]

Its traces are the pedals of the deLongchamps point \( L_o \), the reflection of \( H \) in \( O \).
10.4 Equal-parallelians point

Given triangle $ABC$, we want to construct a point $P$ the three lines through which parallel to the sides cut out equal intercepts. Let $P = xA + yB + zC$ in absolute barycentric coordinates. The parallel to $BC$ cuts out an intercept of length $(1 - x)a$. It follows that the three intercepts parallel to the sides are equal if and only if

$$1 - x : 1 - y : 1 - z = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}.$$ 

The right hand side clearly gives the homogeneous barycentric coordinates of $I^*$, the isotomic conjugate of the incenter $I$. \(^1\) This is a point we can easily construct. Now, translating into *absolute* barycentric coordinates:

$$I^* = \frac{1}{2}[(1 - x)A + (1 - y)B + (1 - z)C] = \frac{1}{2}(3G - P).$$

we obtain $P = 3G - 2I^*$, and can be easily constructed as the point dividing the segment $I^*G$ externally in the ratio $I^*P : PG = 3 : -2$. The point $P$ is called the congruent-parallelians point of triangle $ABC$.

\(^1\)The isotomic conjugate of the incenter appears in ETC as the point $X_{75}$.
Some applications of barycentric coordinates
Chapter 11

Computation of barycentric coordinates

11.1 The Feuerbach point

**Proposition 11.1.** The homogeneous barycentric coordinates of the Feuerbach point are

\[(b + c - a)(b - c)^2 : (c + a - b)(c - a)^2 : (a + b - c)(a - b)^2\].

**Proof.** The Feuerbach point \(F_e\) is the point of (internal) tangency of the incircle and the nine-point circle. It divides \(NI\) in the ratio \(NF_e : F_e I = R^2 : r = R : 2r\). Therefore,

\[F_e = \frac{R \cdot I - 2r \cdot N}{R - 2r}\]

in absolute barycentric coordinates.

From the homogeneous barycentric coordinates of \(N\),

\[(a^2(b^2 + c^2) - (b^2 - c^2)^2, b^2(c^2 + a^2) - (c^2 - a^2)^2, c^2(a^2 + b^2) - (a^2 - b^2)^2) = 32\Delta^2 \cdot N\].

we have

\[2r \cdot N = \frac{r}{16\Delta^2} \left( a^2(b^2 + c^2) - (b^2 - c^2)^2, \ldots \right)\]

\[= \frac{R}{4sabc} \left( a^2(b^2 + c^2) - (b^2 - c^2)^2, \ldots \right),\]

\[R \cdot I = \frac{R}{2s}(a, b, c) = \frac{R}{4sabc} \cdot 2abc(a, b, c).\]
Therefore,

\[ F_e \sim R \cdot I - 2r \cdot N \]
\[ \sim 2abc \cdot a - (a^2(b^2 + c^2) - (b^2 - c^2)^2) \]
\[ = (a^2(2bc - b^2 - c^2) + (b - c)^2(b + c)^2, \ldots, \ldots) \]
\[ = ((b - c)^2 + (b - c)^2(b + c)^2, \ldots, \ldots) \]
\[ = (a + b + c)(b + c - a)(b - c)^2, \ldots, \ldots) \]
\[ \sim ((b + c - a)(b - c)^2, \ldots, \ldots). \]

\[ \square \]

**Proposition 11.2.** (a) \( ON_a \) is parallel to \( NF_e \).
(b) \( ON_a = R - 2r \).
(c) \( N_aH \) is parallel to \( OI \).
(d) The reflection of \( H \) in \( I \) and the reflection of \( N_a \) in \( O \) are the same point.
(e) \( IN \) and \( OS_p \) intersect at the midpoint of \( N_aH \).
11.2 The $OI$ line

11.2.1 The circumcenter of the excentral triangle

Let $I'$ be the reflection of $I$ in $O$. Show that $I'$ is also the midpoint of $N_aL_o$.

Let $N'$ be the midpoint of $O L_o$. The triangles $O I'N'$ and $O I N$ are congruent. $I' N'$ is parallel to $I N$ and hence $N_a O$. Furthermore, $I' N' = I N = \frac{1}{2} N_a O$. It follows that $I'$ is the midpoint of $N_a L_o$.

\[ I' = (a (a^3 + a^2 (b + c) - a (b + c)^2 - (b + c) (b - c)^2) : \cdots : \cdots). \]

11.2.2 The centers of similitude of the circumcircle and the incircle

\[ T_+ = (a^2 (s - a) : b^2 (s - b) : c^2 (s - c)). \]

Proof.

\[
T_+ \sim r \cdot O + R \cdot I \\
= \frac{r}{16 \Delta} (a^2 (b^2 + c^2 - a^2), b^2 (c^2 + a^2 - b^2), c^2 (a^2 + b^2 - c^2)) + \frac{R}{2s} (a, b, c) \\
\sim (a^2 (b^2 + c^2 - a^2), b^2 (c^2 + a^2 - b^2), c^2 (a^2 + b^2 - c^2)) + 8 R R s (a, b, c) \\
= (a^2 (b^2 + c^2 - a^2), b^2 (c^2 + a^2 - b^2), c^2 (a^2 + b^2 - c^2)) + 2 a b c (a, b, c) \\
= (a^2 (b^2 + 2 b c + c^2 - a^2), \cdots, \cdots) \\
= (a^2 (a + b + c) (b + c - a), \cdots, \cdots) \\
\sim (a^2 (b + c - a), \cdots, \cdots). \]

\[ \square \]

\[ T_- = \left( \frac{a^2}{s-a} : \frac{b^2}{s-b} : \frac{c^2}{s-c} \right). \]

Proof.

\[
T_- \sim r \cdot O - R \cdot I \\
= \frac{r}{16 \Delta} (a^2 (b^2 + c^2 - a^2), b^2 (c^2 + a^2 - b^2), c^2 (a^2 + b^2 - c^2)) - \frac{R}{2s} (a, b, c) \\
\sim (a^2 (b^2 + c^2 - a^2), b^2 (c^2 + a^2 - b^2), c^2 (a^2 + b^2 - c^2)) - 8 R R s (a, b, c) \\
= (a^2 (b^2 + c^2 - a^2), b^2 (c^2 + a^2 - b^2), c^2 (a^2 + b^2 - c^2)) - 2 a b c (a, b, c) \\
= (a^2 (b^2 + 2 b c + c^2 - a^2), \cdots, \cdots) \\
= (a^2 (b + a - c) (b + c - a), \cdots, \cdots) \\
\sim (a^2 (c + a - b) (a + b - c), \cdots, \cdots). \]

\[ \square \]
Example 11.1. (a) \(G, T_+, F_e\) are collinear.  
(b) \(H, T_-, F_e\) are collinear.

Proof. (a) This follows from

\[
\begin{align*}
(a^2(b + c - a), b^2(c + a - b), c^2(a + b - c)) \\
- ((b + c - a)(b - c)^2, (c + a - b)(c - a)^2, (a + b - c)(a - b)^2) \\
= ((b + c - a)(a^2 - (b - c)^2), (c + a - b)(b^2 - (c - a)^2), (a + b - c)(c^2 - (a - b)^2)) \\
= ((b + c - a)(a - b + c)(a + b - c), (c + a - b)(b - c + a)(b + c - a), \\
(a + b - c)(c - a + b)(c + a - b)) \\
= (b + c - a)(c + a - b)(a + b - c)(1, 1, 1).
\end{align*}
\]

\[\square\]

11.2.3 The homothetic center \(T\) of excentral and intouch triangles

The two triangles are homothetic since their corresponding sides are perpendicular to the angle bisectors of triangle \(ABC\). Denote by \(T\) the homothetic center. This is clearly the exsimilicenter of their circumcircles. It is therefore the point dividing \(I'\) and \(I\) in the ratio \(I'T : IT = 2R : r\). It follows that \(OT : TI = 2R + r : -2r\).

\[
\begin{array}{c}
T' \\
O \\
T
\end{array}
\]

\[
T = \left( \frac{a}{s-a} : \frac{b}{s-b} : \frac{c}{s-c} \right).
\]

Proof.

\[
T \sim -2r \cdot O + (2R + r)I \\
= \frac{-2r}{16r^2s^2} (a^2(b^2 + c^2 - a^2), \ldots, \ldots) + \frac{2R + r}{2s}(a, b, c) \\
\sim -(a^2(b^2 + c^2 - a^2), \ldots, \ldots) + (2R + r)(4rs)(a, b, c) \\
= -(a^2(b^2 + c^2 - a^2), \ldots, \ldots) + (2abc + 4(s - a)(s - b)(s - c))(a, b, c) \\
= (a(-a(b^2 + c^2 - a^2) + (2abc + 4(s - a)(s - b)(s - c))), \ldots, \ldots) \\
= \left( \frac{1}{2}a(a + b + c)(c + a - b)(a + b - c), \ldots, \ldots \right) \\
\sim \left( \frac{a}{b + c - a}, \ldots, \ldots \right).
\]

\[\square\]
11.3 The excentral triangle

Exercise

1. Show that $TI : IO = 2r : 2R - r$.

2. Find the ratio of division $T_4T : TT_1$.

3. Show that $G, G_e, \text{and } T$ are collinear by finding $p, q$ satisfying

$$
p(1, 1, 1) + q((c + a - b)(a + b - c), (a + b - c)(b + c - a), (b + c - a)(c + a - b)) = 2(a + c - b)(a + b - c), b(a + b - c)(b + c - a), c(b + c - a)(c + a - b)).
$$

Answer: $p = -(b + c - a)(c + a - b)(a + b - c)$ and $q = a + b + c$.

4. Given that $GG_e : G_eT = 2(2R - r) : 3r$, show that $G_e, I, L_o$ are collinear.

Apply the converse of Menelaus’ theorem to triangle $OGT$ with $G_e$ on $GT$, $I$ on $TO$, and $L_o$ on $OG$.

$$\frac{GG_e}{G_eT} \cdot \frac{TI}{IO} \cdot \frac{OL_o}{L_oG} = \frac{2(2R - r)}{3r} \cdot \frac{2r}{2R - r} \cdot \frac{-3}{4} = -1.$$

11.3 The excentral triangle

11.3.1 The centroid

The centroid of the excentral triangle is the point

$$\frac{I_a + I_b + I_c}{3} = \frac{1}{3} \left( \frac{(-a, b, c)}{b + c - a} + \frac{(a, -b, c)}{c + a - b} + \frac{(a, b, -c)}{a + b - c} \right)$$

$$\sim (c + a - b)(a + b - c)(-a, b, c) + (a + b - c)(b + c - a)(a, -b, c)$$

$$+ (b + c - a)(c + a - b)(a, b, -c)$$

$$\sim \frac{(a(-s - b)(s - c) + (s - c)(s - a) + (s - a)(s - b))}{(a^2 - 2as - bc + ca + ab), \ldots, \ldots}$$

$$\sim \frac{(a(s^2 - a^2 - bc), \ldots, \ldots)}{(a(-3a^2 + 2a(b + c) + (b - c)^2), \ldots, \ldots)}.$$

11.3.2 The incenter

$$\sim \frac{\cos A}{2} \cdot \frac{(-a, b, c)}{b + c - a} + \frac{\cos B}{2} \cdot \frac{(a, -b, c)}{c + a - b} + \frac{\cos C}{2} \cdot \frac{(a, b, -c)}{a + b - c}$$

$$\sim \frac{\cos A}{2r \cot} \frac{1}{2} (-a, b, c) + \frac{\cos B}{2r \cot} \frac{1}{2} (a, -b, c) + \frac{\cos C}{2r \cot} \frac{1}{2} (a, b, -c)$$

$$\sim \sin A \frac{1}{2} (-a, b, c) + \sin B \frac{1}{2} (a, -b, c) + \sin C \frac{1}{2} (a, b, -c)$$

$$\sim \left( a \left( -\sin A \frac{1}{2} + \frac{B}{2} \sin \frac{B}{2} + \sin \frac{C}{2} \right), \ldots, \ldots \right).$$
Chapter 12

Some interesting circles

12.1 A fundamental principle on 6 concyclic points

12.1.1 The radical axis of two circles

Given two nonconcentric circles $C_1$ and $C_2$. The locus of points of equal powers with respect to the circle is a straight line perpendicular to the line joining their centers. In fact, if the circles are concentric, there is no finite point with equal powers with respect to the circles. On the other hand, if the centers are distinct points $A$ and $B$ at a distance $d$ apart, there is a unique point $P$ with distances $AP = x$ and $PB = d - x$ such that

$$r_1^2 - x^2 = r_2^2 - (d - x)^2.$$

If this common value is $m$, then every point $Q$ on the perpendicular to $AB$ at $P$ has power $m - PQ^2$ with respect to each of the circles. This line is called the radical axis of the two circles.

If the two circles intersect at two distinct points, then the radical axis is the line joining these common points. If the circles are tangent to each other, then the radical axis is the common tangent.

Theorem 12.1. Given three circles with distinct centers, the radical axes of the three pairs of circles are either concurrent or are parallel.

Proof. (1) If any two of the circles are concentric, there is no finite point with equal powers with respect to the three circles.

(2) If the centers of the circles are distinct and noncollinear, then two of the radical axes, being perpendiculars to two distinct lines with a common point, intersect at a point. This intersection has equal powers with respect to all three circles, and also lies on the third radical axis.

(3) If the three centers are distinct but collinear, then the three radical axes three parallel lines, which coincide if any two of them do. This is the case if and only if the three circles two points in common, or at mutually tangent at a point. In this case we say that the circles are coaxial.
12.1.2 Test for 6 concyclic points

**Proposition 12.2.** Let $X, X'$ be points on the sideline $a$, $Y, Y'$ on $b$, and $Z, Z'$ on $c$. The six points are on a circle if and only if the four points on each pair of sidelines are concyclic.

![Diagram of the proposition](image)

**Proof.** It is enough to prove the sufficiency part. Let $\mathcal{C}_a$ be the circle through the four points $Y, Y', Z, Z'$ on $b$ and $c$, and $\mathcal{C}_b$ the one through $Z, Z', X, X'$ (on $c$ and $a$), and $\mathcal{C}_c$ through $X, X', Y, Y'$ (on $a$ and $b$). We claim that these three circles are identical. If not, then they are pairwise distinct. The three pairs among them have radical axes $a$ (for $\mathcal{C}_b$ and $\mathcal{C}_c$), $b$ (for $\mathcal{C}_c$ and $\mathcal{C}_a$), and $c$ (for $\mathcal{C}_a$ and $\mathcal{C}_b$) respectively. Now, the three radical axes of three distinct circles either intersect at a common point (the radical center), or are parallel (when their centers are on a line), or coincide (when, in addition, the three circles are coaxial). In no case can the three radical axes form a triangle (with sidelines $a, b, c$). This shows that the three circles coincide. \qed
12.2 The Taylor circle

Consider the orthic triangle \( XYZ \), and the pedals of each of the points \( X, Y, Z \) on the two sides not containing it. Thus,

<table>
<thead>
<tr>
<th>Sideline ( a )</th>
<th>Pedals of ( X )</th>
<th>Pedals of ( Y )</th>
<th>Pedals of ( Z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>( X_b )</td>
<td>( X_c )</td>
<td></td>
</tr>
<tr>
<td>( b )</td>
<td>( Y_a )</td>
<td>( Y_c )</td>
<td></td>
</tr>
<tr>
<td>( c )</td>
<td>( Z_a )</td>
<td>( Z_b )</td>
<td></td>
</tr>
</tbody>
</table>

It is easy to write down the lengths of various segments. From these we easily determine the coordinates of these pedals. For example, from \( Y_aC = b \cos^2 \gamma \), we have \( AY_a = b - b \cos^2 \gamma = b \sin^2 \gamma \). Note that

\[
AY_a \cdot AY_c = (b - b \cos^2 \gamma)(b \cos^2 \alpha) = b^2 \cos^2 \alpha \sin^2 \gamma = 4R^2 \cos^2 \alpha \sin^2 \beta \sin^2 \gamma,
\]
\[
AZ_a \cdot AZ_b = (c - c \cos^2 \beta)(c \cos^2 \alpha) = c^2 \cos^2 \alpha \sin^2 \beta = 4R^2 \cos^2 \alpha \sin^2 \beta \sin^2 \gamma,
\]
giving \( AY_a \cdot AY_c = AZ_a \cdot AZ_b = \frac{S_{\alpha \beta}}{4R^2} = \frac{S_{\alpha \beta}}{a^2 b^2 c^2} \). Similarly, \( BX_b \cdot BX_c = BZ_b \cdot BZ_a = \frac{S_{\alpha \beta}}{a^2 b^2 c^2} \) and \( CY_c \cdot CY_a = CX_c \cdot CX_b = \frac{S_{\alpha \gamma}}{a^2 b^2 c^2} \). By Proposition 12.2, the six points \( X_b, X_c, Y_c, Y_a, Z_a, Z_b \) are concyclic. The circle containing them is called the Taylor circle.

Exercise

1. Calculate the length of \( X_bX_c \).
2. Find the equation of the line \( X_bX_c \). ¹
3. The three lines \( X_bX_c, Y_cY_a, Z_aZ_b \) bound a triangle with perspector \( K \).

¹ \( -S^2 x + S_{BB} y + S_{CC} z = 0 \).
12.3 Two Lemoine circles

12.3.1 The first Lemoine circle

Given triangle $ABC$, how can one choose a point $K$ so that when parallel lines are constructed through it to intersect each sideline at two points, the resulting six points are on a circle?

![Diagram of triangle ABC with points K, L, Y, Z, and segments XcYc, XbZb, and BXc, CYc.](image)

**Analysis.** By the intersecting chords theorem, $AY_a \cdot AY_c = AZ_a \cdot AZ_b$. We have

$$b^2 \cdot \frac{AY_a}{AC} \cdot \frac{AY_c}{AC} = c^2 \cdot \frac{AZ_a}{AB} \cdot \frac{AZ_b}{AB}$$

$$\Rightarrow b^2 \cdot \frac{AY_c}{AC} = c^2 \cdot \frac{AZ_b}{AB}$$

$$\Rightarrow b^2 \cdot \frac{BX_c}{BC} = c^2 \cdot \frac{CX_b}{CB} = c^2 \cdot \frac{X_bC}{BC}$$

$$\Rightarrow \frac{BX_c}{X_bC} = \frac{c^2}{b^2}$$

Similarly, $\frac{BZ_a}{Z_aA} = \frac{a^2}{b^2}$, and

$$\frac{X_cX_b}{KY_a} = \frac{X_bX_c}{KY_a} = \frac{X_cK}{KY_c} = \frac{BZ_a}{Z_bA} = \frac{a^2}{b^2}$$

Therefore, $BX_c : X_cX_b : X_bC = c^2 : a^2 : b^2$.

These proportions determine the points $X_b$ and $X_c$ on $BC$, and subsequently the other points: the parallels $X_cY_c$ and $X_bZ_b$ (to $c$ and $b$ respectively) intersect at $K$, and the parallel through $K$ to $a$ determines the points $Y_a$ and $Z_a$. The point $K$ is called the Lemoine symmedian point of triangle $ABC$, and the circle containing these six points is called the first Lemoine circle.

Denote by $L$ the center of the first Lemoine circle. Note that $L$ lies on the perpendicular bisector of each of the parallel segments $X_cX_b$ and $Z_aY_a$. This means that the line joining the midpoints of these segments is the common perpendicular bisector of the segments, and the trapezoid $X_cX_bY_aZ_a$ is symmetric; so are $Y_aY_cZ_bX_b$ and $Z_bZ_aX_cY_c$ by the same reasoning. It follows that the segments $Y_cZ_b$, $Z_aX_c$, and $X_bY_a$ have equal lengths. (Exercise: Show that this common length is $\frac{ab_c}{a^2+b^2+c^2}$.)
12.3.2 The second Lemoine circle

Now, if we construct the parallels of these segments through $K$ to intersect the sidelines, we obtain another three equal segments with common midpoint $K$. The 6 endpoints, $X'_b$, $X'_c$ on $a$, $Y'_b$, $Y'_c$ on $b$, and $Z'_a$, $Z'_c$ on $c$, are on a circle center $K$ and radius $\frac{abc}{a^2+b^2+c^2}$. This circle is called the second Lemoine circle of triangle $ABC$.

\[ \text{12.3.3 Construction of } K \]

The length of $AK$ is twice the median of triangle $AY_cZ_b$ on the side $Y_cZ_b$. If we denote by $m_a$ etc the lengths of medians of triangle $ABC$, then $AK = \frac{2bcm_a}{a^2+b^2+c^2}$. Similarly, $BK = \frac{2acm_b}{a^2+b^2+c^2}$ and $CK = \frac{2abm_c}{a^2+b^2+c^2}$. This allows us to determine the angles $KAB$ etc and the radii of the circles $KAB$ etc. The radius of the circle $KAB$, for example, is $R_c = \frac{AB \cdot AK \cdot BK}{4\Delta(KAB)} = \frac{abcm_a m_b}{(a^2+b^2+c^2)\Delta(ABC)}$. It follows that $\sin KAB = \frac{BK}{2R_c} = \frac{\Delta}{bm_a}$.

Consider also the circle $GAC$. This has radius $R'_a = \frac{AC \cdot AG \cdot GC}{4\Delta(GAC)} = \frac{bma m_a}{3\Delta}$. From this, $\sin GAC = \frac{CG}{2R'_a} = \frac{\Delta}{bm_a}$. This shows that $\angle KAB = \angle GAC$, and $AK$ and the median $AG$ are isogonal lines with respect to the sides $AB$ and $AC$. Similarly, $BK$ and $CK$ are the lines isogonal to the medians $BG$ and $CG$ respectively, and $K$ is the symmedian point of triangle $ABC$. 
12.3.4 The center of the first Lemoine circle

We show that the center $L$ of the first Lemoine circle is the midpoint between $K$ and the circumcenter $O$. It is clear that this center is on the perpendicular bisector of $X_bX_c$. Let $M$ be the midpoint of $X_bX_c$, and $X$ the orthogonal projection of $K$ on $BC$.

We have

$$BM = \frac{ac^2}{a^2 + b^2 + c^2} + \frac{1}{2} \cdot \frac{a^3}{a^2 + b^2 + c^2} = \frac{a(a^2 + 2c^2)}{2(a^2 + b^2 + c^2)},$$

and

$$BX = BX_c + KX_c \cos B = \frac{ac^2}{a^2 + b^2 + c^2} + \frac{a^2c}{a^2 + b^2 + c^2} \cdot \frac{c^2 + a^2 - b^2}{2ca}$$

$$= \frac{ac^2 + a(c^2 + a^2 - b^2)}{2(a^2 + b^2 + c^2)} = \frac{a(a^2 - b^2 + 3c^2)}{2(a^2 + b^2 + c^2)}.$$

It follows that

$$BX + BD = \frac{a(a^2 - b^2 + 3c^2)}{2(a^2 + b^2 + c^2)} + \frac{a(a^2 + b^2 + c^2)}{2(a^2 + b^2 + c^2)} = \frac{a(a^2 + 2c^2)}{a^2 + b^2 + c^2} = 2 \cdot BM.$$

This means that $M$ is the midpoint of $XD$, and the perpendicular to $a$ at $M$ contains the midpoint of $OK$. The same reasoning shows that the midpoint of $OK$ also lies on the perpendiculars to $b$ and $c$ respectively at the midpoints of $Y_cY_a$ and $Z_aZ_b$. It is therefore the center $L$ of the first Lemoine circle. (Exercise. Calculate the radius of the first Lemoine circle).
Chapter 13

Straight line equations

13.1 Area and barycentric coordinates

Theorem 13.1. If for \( i = 1, 2, 3 \), \( P_i = x_i \cdot A + y_i \cdot B + z_i \cdot C \) (in absolute barycentric coordinates), then the area of the oriented triangle \( P_1P_2P_3 \) is

\[
\Delta P_1P_2P_3 = \begin{vmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  x_3 & y_3 & z_3 \\
\end{vmatrix} \cdot \Delta ABC.
\]

Example 13.1. (Area of cevian triangle) Let \( P = (u : v : w) \) be a point with cevian triangle \( XYZ \). The area of the cevian triangle \( XYZ \) is

\[
\frac{1}{(v+w)(w+u)(u+v)} \begin{vmatrix}
  0 & v & w \\
  u & 0 & w \\
  u & v & 0 \\
\end{vmatrix} \cdot \Delta = \frac{2uvw}{(v+w)(w+u)(u+v)} \cdot \Delta.
\]

If \( P \) is an interior point (so that \( u, v, w \) are positive), then \( v + w \geq 2\sqrt{vw} \), \( w + u \geq 2\sqrt{wu} \), and \( u + v \geq 2\sqrt{uv} \). It follows that

\[
\frac{2uvw}{(v+w)(w+u)(u+v)} \leq \frac{2uvw}{2\sqrt{vw} \cdot 2\sqrt{wu} \cdot 2\sqrt{uv}} = \frac{1}{4}.
\]

Equality holds if and only if \( u = v = w \), i.e., \( P = (1 : 1 : 1) = G \), the centroid. The centroid is the interior point with largest cevian triangle.
13.2 Equations of straight lines

13.2.1 Two-point form

The area formula has an easy and extremely important consequence: three points \( P_i = (u_i, v_i, w_i) \) are collinear if and only if

\[
\begin{vmatrix}
  u_1 & v_1 & w_1 \\
  u_2 & v_2 & w_2 \\
  u_3 & v_3 & w_3
\end{vmatrix} = 0.
\]

Consequently, the equation of the line joining two points with coordinates \((x_1 : y_1 : z_1)\) and \((x_2 : y_2 : z_2)\) is

\[
\begin{vmatrix}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  x & y & z
\end{vmatrix} = 0,
\]

or

\[
(y_1 z_2 - y_2 z_1)x + (z_1 x_2 - z_2 x_1)y + (x_1 y_2 - x_2 y_1)z = 0.
\]

Examples

1. The equations of the sidelines \( BC, CA, AB \) are respectively \( x = 0, y = 0, z = 0 \).

2. Given a point \( P = (u : v : w) \), the cevian line \( AP \) has equation \( wy - vz = 0 \); similarly for the other two cevian lines \( BP \) and \( CP \). These lines intersect corresponding sidelines at the traces of \( P \):

\[
A_P = (0 : v : w), \quad B_P = (u : 0 : w), \quad C_P = (u : v : 0).
\]

3. The equation of the line joining the centroid and the incenter is

\[
\begin{vmatrix}
  1 & 1 & 1 \\
  a & b & c \\
  x & y & z
\end{vmatrix} = 0,
\]

or \((b - c)x + (c - a)y + (a - b)z = 0\).

4. The equations of some important lines:

<table>
<thead>
<tr>
<th>Line Type</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler line</td>
<td>( OH = \sum_{cyclic} (b^2 - c^2)(b^2 + c^2 - a^2)x = 0 )</td>
</tr>
<tr>
<td>OI-line</td>
<td>( OI = \sum_{cyclic} bc(b - c)(b + c - a)x = 0 )</td>
</tr>
<tr>
<td>Soddy line</td>
<td>( IG_e = \sum_{cyclic} (b - c)(s - a)^2x = 0 )</td>
</tr>
<tr>
<td>Brocard axis</td>
<td>( OK = \sum_{cyclic} b^2c^2(b^2 - c^2)x = 0 )</td>
</tr>
<tr>
<td>van Aubel line</td>
<td>( HK = \sum_{cyclic} S_{\alpha\alpha}(S_{\beta} - S_{\gamma})x = 0 )</td>
</tr>
</tbody>
</table>
13.2 Equations of straight lines

13.2.2 Intersection of two lines

The intersection of the two lines

\[ p_1x + q_1y + r_1z = 0, \]
\[ p_2x + q_2y + r_2z = 0 \]

is the point

\[ (q_1r_2 - q_2r_1 : r_1p_2 - r_2p_1 : p_1q_2 - p_2q_1). \]

**Proposition 13.2.** Three lines \( p_i x + q_i y + r_i z = 0, \ i = 1, 2, 3, \) are concurrent if and only if

\[
\begin{vmatrix}
 p_1 & q_1 & r_1 \\
 p_2 & q_2 & r_2 \\
 p_3 & q_3 & r_3 \\
\end{vmatrix} = 0.
\]

**Examples**

1. The intersection of the Euler line and the Soddy line is the point

\[
\begin{vmatrix}
 c-a & (s-b)^2 & (s-c)^2 \\
 (c^2-a^2)(c^2+a^2-b^2) & (a-b)(s-c)^2 \\
 (a^2-b^2)(a^2+b^2-c^2) & : & : & : \\
\end{vmatrix} = (c-a)(a-b)
\begin{vmatrix}
 s-b & s-c \\
 (c+a)(c^2+a^2-b^2) & (a+b)(a^2+b^2-c^2) \\
\end{vmatrix} = (c-a)(a-b)
\begin{vmatrix}
 s-b & a-b \\
 (c+a)(c^2+a^2-b^2) & (a+b+c) \\
\end{vmatrix} = (b-c)(c-a)(a-b)
\begin{vmatrix}
 s-b & 4a \\
 (c+a)(c^2+a^2-b^2) & (a+b+c) \\
\end{vmatrix} = \frac{1}{4}(b-c)(c-a)(a-b)(-3a^4 + 2a^2(b^2+c^2) + (b^2-c^2)^2) : : : :
\]

Writing \( a^2 = S_\beta + S_\gamma \) etc. we have

\[
-3a^4 + 2a^2(b^2+c^2) + (b^2-c^2)^2 = -3(S_\beta + S_\gamma)^2 + 2(S_\beta + S_\gamma)(2S_\alpha + S_\beta + S_\gamma) + (S_\beta - S_\gamma)^2 = 4(S_\alpha + S_\gamma - S_\beta).
\]

This intersection has homogeneous barycentric coordinates

\[-S_\beta + S_\gamma + S_\alpha : S_\beta - S_\gamma + S_\alpha : S_\beta + S_\gamma - S_\alpha.\]

This is the reflection of \( H \) in \( O \), and is called the deLongchamps point \( L_o \).
13.3 Perspective triangles

Many interesting points and lines in triangle geometry arise from the *perspectivity* of triangles. We say that two triangles $X_1Y_1Z_1$ and $X_2Y_2Z_2$ are perspective, $X_1Y_1Z_1 \cap X_2Y_2Z_2$, if the lines $X_1X_2$, $Y_1Y_2$, $Z_1Z_2$ are concurrent. The point of concurrency, $\wedge(X_1Y_1Z_1, X_2Y_2Z_2)$, is called the *perspector*. Along with the perspector, there is an *axis of perspectivity*, or the *perspectrix*, which is the line joining

$$Y_1Z_2 \cap Z_1Y_2, \quad Z_1X_2 \cap X_1Z_2, \quad X_1Y_2 \cap Y_1X_2.$$ 

We denote this line by $L_{\wedge}(X_1Y_1Z_1, X_2Y_2Z_2)$.

Homothetic triangles are clearly prespective. If triangles $T$ and $T'$, their perspector is the homothetic center, which we shall denote by $\wedge_0(T, T')$.

**Proposition 13.3.** A triangle with vertices

$$X = U : v : w,$$

$$Y = u : V : w,$$

$$Z = u : v : W,$$

for some $U, V, W$, is perspective to $ABC$ at $\wedge(XYZ) = (u : v : w)$. The perspectrix is the line

$$\frac{x}{u - U} + \frac{y}{v - V} + \frac{z}{w - W} = 0.$$

**Proof.** The line $AX$ has equation $wy - vz = 0$. It intersects the sideline $BC$ at the point $(0 : v : w)$. Similarly, $BY$ intersects $CA$ at $(u : 0 : w)$ and $CZ$ intersects $AB$ at $(u : v : 0)$. These three are the traces of the point $(u : v : w)$.

The line $YZ$ has equation $-(vW - VW)x + u(w - W)y + u(v - V)z = 0$. It intersects the sideline $BC$ at $(0 : v - V : -(w - W))$. Similarly, the lines $ZX$ and $XY$ intersect $CA$ and $AB$ respectively at $-(u - U) : 0 : w - W)$ and $(u - U : -(v - V) : 0)$. It is easy to see that these three points are collinear on the line

$$\frac{x}{u - U} + \frac{y}{v - V} + \frac{z}{w - W} = 0.$$ 

$\square$
The excentral triangle

The excentral triangle is perspective with $ABC$; the perspector is the incenter $I$:

\[
\begin{align*}
I_a &= -a : b : c \\
I_b &= a : -b : c \\
I_c &= a : b : -c \\
I &= a : b : c
\end{align*}
\]

13.3.1 The Conway configuration

Given triangle $ABC$, extend

(i) $CA$ and $BA$ to $Y_a$ and $Z_a$ such that $AY_a = AZ_a = a$,
(ii) $AB$ and $CB$ to $Z_b$ and $X_b$ such that $BZ_b = BX_b = b$,
(iii) $BC$ and $AC$ to $X_c$ and $Y_c$ such that $CX_c = CY_c = c$.

These points have coordinates

\[
\begin{align*}
Y_a &= (a + b : 0 : -a), & Z_a &= (c + a : -a : 0) \\
Z_b &= (-b : b + c : 0), & X_b &= (0 : a + b : -b) \\
X_c &= (0 : -c : c + a), & Y_c &= (-c : 0 : b + c)
\end{align*}
\]

From the coordinates of $Y_c$ and $Z_b$, we determine easily the coordinates of $X = BY_c \cap CZ_b$:

\[
\begin{align*}
Y_c &= -c : 0 : b + c = -bc : 0 : b(b + c) \\
Z_b &= -b : b + c : 0 = -bc : c(b + c) : 0 \\
X &= -bc : c(b + c) : b(b + c)
\end{align*}
\]
Similarly, the coordinates of \( Y = CZ_a \cap AX_c \), and \( Z = AX_b \cap BY_a \) can be determined. The following table shows that the perspector of triangles \( ABC \) and \( XYZ \) is the point with homogeneous barycentric coordinates \( \left( \frac{1}{a} : \frac{1}{b} : \frac{1}{c} \right) \).

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Y )</th>
<th>( Z )</th>
<th>( ? )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-bc : c(b + c) : b(b + c))</td>
<td>( c(c + a) : -ca : a(c + a))</td>
<td>( b(a + b) : a(a + b) : -ab)</td>
<td>( = \frac{1}{a} : \frac{1}{b} : \frac{1}{c} )</td>
</tr>
<tr>
<td>(-bc : c(b + c) : b(b + c))</td>
<td>( c(c + a) : -ca : a(c + a))</td>
<td>( b(a + b) : a(a + b) : -ab)</td>
<td>( = \frac{1}{a} : \frac{1}{b} : \frac{1}{c} )</td>
</tr>
</tbody>
</table>

\( ? = \frac{1}{a} : \frac{1}{b} : \frac{1}{c} \)
13.4 Perspectivity

Many interesting points and lines in triangle geometry arise from the perspectivity of triangles. We say that two triangles \(X_1Y_1Z_1\) and \(X_2Y_2Z_2\) are perspective, \(X_1Y_1Z_1 \sim X_2Y_2Z_2\), if the lines \(X_1X_2, Y_1Y_2, Z_1Z_2\) are concurrent. The point of concurrency, \(\wedge(X_1Y_1Z_1, X_2Y_2Z_2)\), is called the perspector. Along with the perspector, there is an axis of perspectivity, or the perspectrix, which is the line joining containing

\[
Y_1Z_2 \cap Z_1Y_2, \quad Z_1X_2 \cap X_1Z_2, \quad X_1Y_2 \cap Y_1X_2.
\]

We denote this line by \(L_\wedge(X_1Y_1Z_1, X_2Y_2Z_2)\). We justify this in § below.

If one of the triangles is the triangle of reference, it shall be omitted from the notation. Thus, \(\wedge(XYZ) = \wedge(ABC, XYZ)\) and \(L_\wedge(XYZ) = L_\wedge(ABC, XYZ)\).

Homothetic triangles are clearly prespective. If triangles \(T\) and \(T'\), their perspector is the homothetic center, which we shall denote by \(\wedge_0(T, T')\).

**Proposition 13.4.** A triangle with vertices

\[
X = U : v : w, \\
Y = u : V : w, \\
Z = u : v : W,
\]

for some \(U, V, W\), is perspective to \(ABC\) at \(\wedge(XYZ) = (u : v : w)\). The perspectrix is the line

\[
\frac{x}{u-U} + \frac{y}{v-V} + \frac{z}{w-W} = 0.
\]

**Proof.** The line \(AX\) has equation \(wy - vz = 0\). It intersects the sideline \(BC\) at the point \((0 : v : w)\). Similarly, \(BY\) intersects \(CA\) at \((u : 0 : w)\) and \(CZ\) intersects \(AB\) at \((u : v : 0)\). These three are the traces of the point \((u : v : w)\).

The line \(YZ\) has equation \(-(vw - VW)x + u(w - W)y + u(v - V)z = 0\). It intersects the sideline \(BC\) at \((0 : v - V : -(w - W))\). Similarly, the lines \(ZX\) and \(XY\) intersect \(CA\) and \(AB\) respectively at \((-u - U : 0 : w - W)\) and \((u - U : -(v - V) : 0)\). These three points are collinear on the trilinear polar of \((u - U : v - V : w - W)\).

The triangles \(XYZ\) and \(ABC\) are homothetic if the perspectrix is the line at infinity.
13.4.1 The Schiffler point: intersection of four Euler lines

**Theorem 13.5.** Let $I$ be the incenter of triangle $ABC$. The Euler lines of the triangles $IBC$, $ICA$, $IAB$ are concurrent at a point on the Euler line of $ABC$, namely, the Schiffler point

$$S_c = \left( \frac{a(b + c - a)}{b + c} : \frac{b(c + a - b)}{c + a} : \frac{c(a + b - c)}{a + b} \right).$$

**Proof.** Let $I$ be the incenter of triangle $ABC$.

We first compute the equation of the Euler line of the triangle $IBC$.

The centroid of triangle $IBC$ is the point $(a : a + 2b + c : a + b + 2c)$. The circumcenter of triangle is the midpoint of $II_a$. This is the point $(-a^2 : b(b + c) : c(b + c))$. From these we obtain the equation of the Euler line:

$$0 = \begin{vmatrix} x & y & z \\ a & a + 2b + c & a + b + 2c \\ -a^2 & b(b + c) & c(b + c) \end{vmatrix} = (b - c)(b + c)x + a(c + a)y - a(a + b)z.$$

The equations of the Euler lines of the triangles $ICA$ and $IAB$ can be obtained by cyclic permutations of $a$, $b$, $c$ and $x$, $y$, $z$. Thus the three Euler lines are

$$(b - c)(b + c)x + a(c + a)y - a(a + b)z = 0,$$

$$-b(b + c)x + (c - a)(c + a)y + b(a + b)z = 0,$$

$$c(b + c)x - c(c + a)y + (a - b)(a + b)z = 0.$$

Computing the intersection of the latter two lines, we have the point

$$\begin{vmatrix} b - c & (c + a) & (a + b) \\ -b & b & a + b \\ c & a - b & c - a \end{vmatrix} = (c + a)(a - b) + bc : b(c + a - b) : c(b - (c - a)) = a(b + c - a) : b(c + a - b) : c(a + b - c).$$

It is easy to verify that this point also lies on the Euler line of $IBC$ given by the first equation:

$$\begin{align*}
(b - c)a(b + c - a) + ab(c + a - b) - ac(a + b - c) \\
= a((b - c)(b + c - a) + b(c + a - b) - c(a + b - c)) \\
= a(b^2 - c^2 - ab + ca + bc + ab - b^2 - ca - bc + c^2) \\
= 0.
\end{align*}$$

It is routine to verify that this point also lies on the Euler line of $ABC$, with equation

$$(b^2 - c^2)(b^2 + c^2 - a^2)x + (c^2 - a^2)(c^2 + a^2 - b^2)y + (a^2 - b^2)(a^2 + b^2 - c^2)z = 0.$$

This shows that the four Euler lines are concurrent.

□
Chapter 14

Cevian nest theorem

14.1 Trilinear pole and polar

14.1.1 Trilinear polar of a point

Given a point $P$ with traces $A_P$, $B_P$, and $C_P$ on the sidelines of triangle $ABC$, let

$$X = B_P C_P \cap BC, \quad Y = C_P A_P \cap CA, \quad Z = A_P B_P \cap AB.$$ 

These points $X$, $Y$, $Z$ lie on a line called the trilinear polar (or simply tripolar) of $P$. 

![Diagram of a triangle with trilinear polar](image)
If \( P = (u : v : w) \), then \( B_P = (u : 0 : w) \) and \( C_P = (u : v : 0) \). The line \( B_P C_P \) has equation

\[-\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0.\]

It intersects the sideline \( BC \) at the point \( X = (0 : v : -w) \).

Similarly, \( A_P = (0 : v : w) \) and the points \( Y, Z \) are

\[Y = (-u : 0 : w), \quad Z = (u : -v : 0).\]

The line containing the three points \( X, Y, Z \) is

\[\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0.\]

This is the tripolar of \( P \).
14.1.2 Tripole of a line

Given a line $\mathcal{L}$ intersecting $BC, CA, AB$ at $X, Y, Z$ respectively, let

\[
A' = BY \cap CZ, \quad B' = CZ \cap AX, \quad C' = AX \cap BY.
\]

The lines $AA', BB', CC'$ are concurrent. The point of concurrency is the tripole $P$ of $\mathcal{L}$.

Clearly $P$ is the tripole of $\mathcal{L}$ if and only if $\mathcal{L}$ is the tripolar of $P$. 
14.2 Anticevian triangles

The vertices of the anticevian triangle of a point \( P = (u : v : w) \) are the harmonic conjugates of \( P \) with respect to the cevian segments \( AA_P, BB_P \) and \( CC_P \), i.e.,

\[
AP : PA_P = -AP_a : P_a A_P;
\]

similarly for \( P_b \) and \( P_c \). This is called the anticevian triangle of \( P \) since \( ABC \) is the cevian triangle \( P_a P_b P_c \). It is also convenient to regard \( P, P_a, P_b, P_c \) as a harmonic quadruple in the sense that any three of the points constitute the harmonic associates of the remaining point.

14.2.1 Construction of anticevian triangle

If the trilinear polar \( L_P \) of \( P \) intersects the sidelines \( BC, CA, AB \) at \( X', Y', Z' \) respectively, then the anticevian triangle \( \text{cev}^{-1}(P) \) is simply the triangle bounded by the lines \( AX', BY', \) and \( CZ' \).
Another construction of anticevian triangle

Here is an alternative construction of $\text{cev}^{-1}(P)$.

Let $A_H B_H C_H$ be the orthic triangle, and $X$ the reflection of $P$ in $a$, then the intersection of the lines $A_H X$ and $OA$ is the harmonic conjugate $P_a$ of $P$ in $AA_P$:

$$\frac{A P_a}{P_a A P} = -\frac{A P}{P A P}.$$

**Proof.** Let $A'$ be the reflection of $A$ in $BC$. Applying Menelaus’ theorem to triangle $A_P A A'$ with transversal $A_H X P_a$, we have

$$\frac{A P_a}{P_a A P} \cdot \frac{A P X}{X A'} \cdot \frac{A' A_H}{A_H A} = -1.$$

This gives

$$\frac{A P_a}{P_a A P} = -\frac{X A'}{A P X} = -\frac{P A}{A P A} = -\frac{A P}{P A P},$$

showing that $P_a$ and $P$ divide $A A_P$ harmonically. \qed
Examples of anticevian triangles

(1) The anticevian triangle of the centroid is the superior triangle, bounded by the lines through the vertices parallel to the opposite sides.

(2) The anticevian triangle of the incenter is the excentral triangle whose vertices are the excenters.

(3) The vertices of the tangential triangle being

$$A' = (-a^2 : b^2 : c^2), \quad B' = (a^2 : -b^2 : c^2), \quad C' = (a^2 : b^2 : -c^2),$$

these clearly form the anticevian triangle of a point with coordinates $$(a^2 : b^2 : c^2)$$, which we call the symmedian point $K$.

(4) The anticevian triangle of the circumcenter. Here is an interesting property of $cev^{-1}(O)$. Let the perpendiculurs to $AC$ and $AB$ at $A$ intersect $BC$ at $A_b$ and $A_c$ respectively. We call $AA_bA_c$ an orthial triangle of $ABC$. The circumcenter of $AA_bA_c$ is the vertex $O_a$ of $cev^{-1}(O)$; similarly for the other two orthial triangles. (See §??).
14.3 Cevian quotients

14.3.1 The cevian nest theorem

Theorem 14.1. For arbitrary points $P$ and $Q$, the cevian triangle $\text{cev}(P)$ and the anticevian triangle $\text{cev}^{-1}(Q)$ are always perspective. If $P = (u : v : w)$ and $Q = (u' : v' : w')$, then

(i) the perspector is the point

$$\wedge(\text{cev}(P), \text{cev}^{-1}(Q)) = \left( u' \left( -\frac{u'}{u} + \frac{v'}{v} + \frac{w'}{w} \right) : v' \left( -\frac{v'}{v} + \frac{w'}{w} + \frac{u'}{u} \right) : w' \left( -\frac{w'}{w} + \frac{u'}{u} + \frac{v'}{v} \right) \right),$$

(ii) the perspectrix is the line $L_{\wedge}(\text{cev}(P), \text{cev}^{-1}(Q))$ with equation

$$\sum_{\text{cyclic}} \frac{1}{u} \left( -\frac{u'}{u} + \frac{v'}{v} + \frac{w'}{w} \right) x = 0.$$

Proof. (i) Let $\text{cev}(P) = XYZ$ and $\text{cev}^{-1}(Q) = X'Y'Z'$. Since $X = (0 : v : w)$ and $X' = (-u' : v' : w')$, the line $XX'$ has equation

$$\frac{1}{u'} \left( \frac{w'}{w} - \frac{v'}{v} \right) x - \frac{1}{v} y + \frac{1}{w} z = 0.$$

The equations of $YY'$ and $ZZ'$ can be easily written down by cyclic permutations of $(u, v, w), (u', v', w')$ and $(x, y, z)$. It is easy to check that the line $XX'$ contains the point

$$\left( u' \left( -\frac{u'}{u} + \frac{v'}{v} + \frac{w'}{w} \right) : v' \left( -\frac{v'}{v} + \frac{w'}{w} + \frac{u'}{u} \right) : w' \left( -\frac{w'}{w} + \frac{u'}{u} + \frac{v'}{v} \right) \right)$$
whose coordinates are invariant under the above cyclic permutations. This point therefore also lies on the lines \( YY' \) and \( ZZ' \).

(ii) The lines \( YZ \) and \( Y'Z' \) have equations

\[
-\frac{x}{u} + \frac{y}{v} + \frac{z}{w} = 0,
\frac{x}{u'} + \frac{y}{v'} + \frac{z}{w'} = 0.
\]

They intersect at the point

\[U' = (u(wv' - uw') : vwv' : -vww').\]

Similarly, the lines pairs \( ZX, Z'X' \) and \( XY, X'Y' \) have intersections

\[V' = (-wuu' : v(ww' - uu') : wuw')\]
and

\[W' = (uvu' : -uvv' : w(vu' - uv')).\]

The three points \( U', V', W' \) lie on the line with equation given above.

**Corollary 14.2.** If \( T' \) is a cevian triangle of \( T \) and \( T'' \) is a cevian triangle of \( T' \), then \( T'' \) is a cevian triangle of \( T \).

**Proof.** With reference to \( T_2 \), the triangle \( T_1 \) is anticevian.

**Remark.** Suppose \( T' = \text{cev}_T(P) \) and \( T'' = \text{cev}_{T'}(Q) \). If \( P = (u : v : w) \) with respect to \( T \), and \( Q = (u' : v' : w') \) with respect to \( T' \), then,

\[\wedge(T, T'') = \left(\frac{u}{u'}(v + w) : \frac{v}{v'}(w + u) : \frac{w}{w'}(u + v)\right)\]

with respect to triangle \( T \). The equation of the perspectrix \( \mathcal{L}_{\wedge}(T, T'') \) is

\[
\sum_{\text{cyclic}} \frac{1}{u} \left(-\frac{v + w}{u'} + \frac{w + u}{v'} + \frac{u + v}{w'}\right) x = 0.
\]

These formulae, however, are quite difficult to use, since they involve complicated changes of coordinates with respect to different triangles.

We shall simply write

\[P/Q := \wedge(\text{cev}(P), \text{cev}^{-1}(Q))\]

and call it the **cevian quotient** of \( P \) by \( Q \).
The cevian quotients of the centroid $G/P$

If $P = (u : v : w)$,

$$G/P = (u(-u + v + w) : v(-v + w + u) : w(-w + u + v)).$$

Some common examples of $G/P$.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$G/P$ coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>$M_i$ $(a(s - a) : b(s - b) : c(s - c))$</td>
</tr>
<tr>
<td>$O$</td>
<td>$K$ $(a^2 : b^2 : c^2)$</td>
</tr>
<tr>
<td>$K$</td>
<td>$O$ $(a^2S_a : b^2S_b : c^2S_c)$</td>
</tr>
</tbody>
</table>

The point $M_i = G/I$ is called the Mittenpunkt of triangle $ABC$.\(^1\) It is the symmedian point of the excentral triangle. The tangential triangle of the excentral triangle is homothetic to $ABC$ at $T$.

(i) We compute the symmedian point of the excentral triangle. Note that

$$I_bI_c^2 = \frac{a^2bc}{(s - b)(s - c)}, \quad I_cI_a^2 = \frac{ab^2c}{(s - c)(s - a)}, \quad I_aI_b^2 = \frac{abc^2}{(s - a)(s - b)}.$$ 

For the homogeneous barycentric coordinates of the symmedian point of the excentral triangle $I_aI_bI_c$, we have

$$I_bI_c^2 \cdot I_a + I_cI_a^2 \cdot I_b + I_aI_b^2 \cdot I_c$$

$$= \frac{a^2bc}{(s - b)(s - c)} \cdot \frac{(-a, b, c)}{2(s - a)} + \frac{ab^2c}{(s - c)(s - a)} \cdot \frac{(a, -b, c)}{2(s - b)} + \frac{abc^2}{(s - a)(s - b)} \cdot \frac{(a, b, -c)}{2(s - c)}$$

$$= \frac{abc}{2(s - a)(s - b)(s - c)} (a(-a, b, c) + b(a, -b, c) + c(a, b, -c))$$

$$= \frac{abc}{2(s - a)(s - b)(s - c)} (a(-a + b + c), b(a - b + c), c(a + b - c)).$$

From this, we obtain the Mittenpunkt $M_i$.

(ii) Since the excentral triangle is homothetic to the intouch triangle at $T$, their tangential triangles are homothetic at the same triangle center. Using the cevian nest theorem, we have

$$T = \bigwedge_0(\text{cev}(G_e), \text{cev}^{-1}(I)) = \left(\frac{a}{s-a} : \frac{b}{s-b} : \frac{c}{s-c}\right).$$

\(^1\)This appears as $X_9$ in ETC.
The cevian quotients of the orthocenter $H/P$

For $P = (u : v : w)$,

$$H/P = (u(-S_\alpha u + S_\beta v + S_\gamma w) : v(-S_\beta v + S_\gamma w + S_\alpha u) : w(-S_\gamma w + S_\alpha u + S_\beta v)).$$

**Examples**

(1) $H/G = (S_\beta + S_\gamma - S_\alpha : S_\gamma + S_\alpha - S_\beta : S_\alpha + S_\beta - S_\gamma)$ is the superior of $H^*$.  

(2) $H/I$ is a point on the $OI$-line, dividing $OI$ in the ratio $R + r : -2r$. \(^2\)

$$H/I = (a(a^3 + a^2(b + c) - a(b^2 + c^2) - (b + c)(b - c)^2) : \cdots : \cdots).$$

(3) $H/K = \left(\frac{a^2}{S_\alpha} : \frac{b^2}{S_\beta} : \frac{c^2}{S_\gamma}\right)$ is the homothetic center of the orthic and tangential triangle. \(^3\) It is a point on the Euler line.

(4) $H/O$ is the orthocenter of the tangential triangle. \(^4\)

(5) $H/N$ is the orthocenter of the orthic triangle. \(^5\)

**The cevian quotient $G_e/K$**

This is the perspector of the intouch triangle and the tangential triangle \(^6\)

$$G_e/K = (a^2(a^3 - a^2(b + c) + a(b^2 + c^2) - (b + c)(b - c)^2) : \cdots : \cdots).$$

\(^2\)This point appears as $X_{46}$ in ETC.  
\(^3\)This appears as $X_{25}$ in ETC.  
\(^4\)This appears as $X_{155}$ in ETC.  
\(^5\)This appears as $X_{52}$ in ETC.  
\(^6\)This appears as $X_{1486}$ in ETC.
14.3.2 Basic properties of cevian quotients

Proposition 14.3. (1) $P/P = P$.

(2) If $P/Q = M$, then $Q = P/M$.

Proof. (2) Let $P = (u : v : w)$, $Q = (u' : v' : w')$, and $M = (x : y : z)$. We have

$$
x = rac{u'}{u} \left( -rac{u'}{u} + rac{v'}{v} + rac{w'}{w} \right),
$$

$$
y = rac{v'}{v} \left( rac{u'}{u} - rac{v'}{v} + rac{w'}{w} \right),
$$

$$
z = rac{w'}{w} \left( rac{u'}{u} + rac{v'}{v} - rac{w'}{w} \right),
$$

From these,

$$
-x + y + z = \left( \frac{u'}{u} - \frac{v'}{v} + \frac{w'}{w} \right) \left( \frac{u'}{u} + \frac{v'}{v} - \frac{w'}{w} \right),
$$

$$
x - y + z = \left( -\frac{u'}{u} + \frac{v'}{v} + \frac{w'}{w} \right) \left( \frac{u'}{u} + \frac{v'}{v} - \frac{w'}{w} \right),
$$

$$
x + y - z = \left( -\frac{u'}{u} - \frac{v'}{v} + \frac{w'}{w} \right) \left( \frac{u'}{u} - \frac{v'}{v} + \frac{w'}{w} \right),
$$

$$
x + y - z = \left( -\frac{u'}{u} - \frac{v'}{v} - \frac{w'}{w} \right) \left( \frac{u'}{u} + \frac{v'}{v} + \frac{w'}{w} \right).
and
\[
\frac{x}{u} \left( \frac{-x + y + z}{u} \right) : \frac{y}{v} \left( \frac{x - y + z}{v} \right) : \frac{z}{w} \left( \frac{x + y - z}{w} \right) = \frac{u'}{u} : \frac{v'}{v} : \frac{w'}{w}.
\]

It follows that
\[
u' : v' : w' = x \left( \frac{-x + y + z}{u} \right) : y \left( \frac{x - y + z}{v} \right) : z \left( \frac{x + y - z}{w} \right).
\]
Chapter 15

Circle equations

15.1 The power of a point with respect to a circle

The power of $P$ with respect to a circle $Q(\rho)$ is the quantity $\mathcal{P}(P) := PQ^2 - \rho^2$. A point $P$ is in, on, or outside the circle according as $\mathcal{P}$ is negative, zero, or positive.

Proposition 15.1. Let $P = xA + yB + zC$ in absolute barycentric coordinates.

$$\mathcal{P}(P) = x\mathcal{P}(A) + y\mathcal{P}(B) + z\mathcal{P}(C) - (a^2yz + b^2zx + c^2xy).$$

Proof. Let $X$ be the trace of $P$ on $BC$. In absolute coordinates, $X = \frac{yB + zC}{y + z}$, so that $P = xA + (y + z)X$. Applying Stewart’s theorem in succession to triangles $QAX$, $QBC$ and $ABC$, we have

$$PQ^2 = xAQ^2 + (y + z)XQ^2 - x(y + z)AX^2$$

$$= xAQ^2 + (y + z) \left( \frac{y}{y + z} BQ^2 + \frac{z}{y + z} CQ^2 - \frac{yz}{(y + z)^2} BC^2 \right) - x(y + z)AX^2$$

$$= xAQ^2 + yBQ^2 + zCQ^2 - \frac{yz}{y + z} BC^2 - x(y + z)AX^2$$

$$= xAQ^2 + yBQ^2 + zCQ^2 - \frac{yz}{y + z} BC^2$$

$$- x(y + z) \left( \frac{z}{y + z} AC^2 + \frac{y}{y + z} AB^2 - \frac{yz}{(y + z)^2} \cdot BC^2 \right)$$

$$= xAQ^2 + yBQ^2 + zCQ^2 - \frac{(1 - x)yz}{y + z} \cdot BC^2 - xz \cdot AC^2 - xy \cdot AB^2$$

$$= xAQ^2 + yBQ^2 + zCQ^2 - (a^2yz + b^2zx + c^2xy).$$
From this it follows that

\[ \mathcal{P}(P) = PQ^2 - \rho^2 \]
\[ = x(AQ^2 - \rho^2) + y(BQ^2 - \rho^2) + z(CQ^2 - \rho^2) - (a^2yz + b^2zx + c^2xy) \]
\[ = x\mathcal{P}(A) + y\mathcal{P}(B) + z\mathcal{P}(C) - a^2yz - b^2zx - c^2xy. \]

\[ \square \]

15.2 Circle equation

Using homogeneous barycentric coordinates for \( P \) and writing

\[ f := \mathcal{P}(A), \quad g := \mathcal{P}(B), \quad h := \mathcal{P}(C), \]

for the powers of \( A, B, C \) with respect to a circle \( Q(\rho) \), we have,

\[ \mathcal{P}(P) = \frac{fx + gy + hz}{x + y + z} - \frac{a^2yz + b^2zx + c^2xy}{(x + y + z)^2} \]
\[ = \frac{(a^2yz + b^2zx + c^2xy) - (x + y + z)(fx + gy + hz)}{(x + y + z)^2}. \]

Therefore, the equation of the circle is

\[ (a^2yz + b^2zx + c^2xy) - (x + y + z)(fx + gy + hz) = 0. \]

**Example 15.1.** (1) The equation of the circumcircle is \( a^2yz + b^2zx + c^2xy = 0 \) since \( f = g = h = 0 \).

(2) For the incircle, we have

\[ f = (s - a)^2, \quad g = (s - b)^2, \quad h = (s - c)^2. \]

The equation of the incircle is

\[ a^2yz + b^2zx + c^2xy - (x + y + z)((s - a)^2x + (s - b)^2y + (s - c)^2z) = 0. \]

(3) Similarly, the \( A \)-excircle has equation

\[ a^2yz + b^2zx + c^2xy - (x + y + z)(s^2x + (s - c)^2y + (s - b)^2z) = 0. \]

(4) For the nine-point circle, we have \( f = \frac{b}{2} \cdot c \cos A = \frac{b}{2} \cdot \frac{S_a}{b} = \frac{1}{2}S_a \). Similarly, \( g = \frac{1}{2}S_\beta \) and \( h = \frac{1}{2}S_\gamma \). Therefore, the equation of the nine-point circle is

\[ 2(a^2yz + b^2zx + c^2xy) - (x + y + z)(S_\alpha x + S_\beta y + S_\gamma z) = 0. \]
Exercise

1. Find the equation of the Conway circle.

2. Find the equations of the circles
   (i) $\mathcal{C}_{BBC}$ passing through $B$ and $C$, and tangent to $BC$ at $B$,
   (ii) $\mathcal{C}_{BCC}$ passing through $B$ and $C$, and tangent to $BC$ at $C$.

3. Compute the coordinates of the Brocard points:
   (i) $\Omega_\to$ as the intersection of the circles $\mathcal{C}_{BBC}$, $\mathcal{C}_{CCA}$, and $\mathcal{C}_{AAB}$,
   (ii) $\Omega_\leftarrow$ as the intersection of the circles $\mathcal{C}_{BCC}$, $\mathcal{C}_{CAA}$, and $\mathcal{C}_{ABB}$.

4. Find the equation of the circle with diameter $BC$.

15.3 Points on the circumcircle

The equation of the circumcircle can be written in the form

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} = 0.$$  

This shows that the circumcircle consists of the isogonal conjugates of infinite points.

15.3.1 $X(101)$

The point

$$X(101) = \left( \frac{a^2}{b-c} : \frac{b^2}{c-a} : \frac{c^2}{a-b} \right)$$

is clearly on the circumcircle.

15.3.2 $X(100)$

The point

$$X(100) = \left( \frac{a}{b-c} : \frac{b}{c-a} : \frac{c}{a-b} \right)$$

is clearly on the circumcircle. It is the isogonal conjugate of the infinite point

$$(a(b-c) : b(c-a) : c(a-b))$$

(on the trilinear polar of the incenter, namely, the line $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$).

Its inferior is a point on the nine-point circle. To find this, we rewrite

$$X(100) = (a(c-a)(a-b) : b(a-b)(b-c) : c(b-c)(c-a)).$$
From this,
\[
\inf(X(100)) = (b(a - b)(b - c) + c(b - c)(c - a) : \cdots : \cdots)
\]
\[
= ((b - c)(b(a - b) + c(c - a)) : \cdots : \cdots)
\]
\[
= ((b - c)^2(b + c - a) : \cdots : \cdots),
\]
the Feuerbach point!

1. The distance from \( X(100) \) to the Nagel point is the diameter of the incircle.
2. \( X(100) \) is the intersection of the Euler lines of the triangles \( I_aBC, I_bCA, I_cAB \).

### 15.3.3 The Steiner point \( X(99) \)

The Steiner point
\[
X(99) = \left( \frac{1}{b^2 - c^2} : \frac{1}{c^2 - a^2} : \frac{1}{a^2 - b^2} \right)
\]

The inferior of the Steiner point is
\[
X(115) = ((b^2 - c^2)^2 : (c^2 - a^2)^2 : (a^2 - b^2)^2).
\]

This is the midpoint between the Fermat points.

### 15.3.4 The Euler reflection point \( E = X(110) \)

**Theorem 15.2.** The reflections of the Euler line in the sidelines of triangle \( ABC \) are concurrent at
\[
E = \left( \frac{a^2}{b^2 - c^2} : \frac{b^2}{c^2 - a^2} : \frac{c^2}{a^2 - b^2} \right)
\]
on the circumcircle.

**Proof.** The Euler line
\[
S_\alpha(S_\beta - S_\gamma)x + S_\beta(S_\gamma - S_\alpha)y + S_\gamma(S_\alpha - S_\beta)z = 0
\]
intersects the sideline \( BC \) at
\[
X = (0 : S_\gamma(S_\alpha - S_\beta) : -S_\beta(S_\gamma - S_\alpha)).
\]

We find the reflection of \( H \) in \( BC \) as follows. From the relation
\[
(S_\beta + S_\gamma + S_\alpha)(0, S_\gamma, S_\beta) + S_\beta(S_\beta + S_\gamma, -S_\gamma, -S_\beta)
\]
\[
= (S_\beta + S_\gamma)(S_\beta + S_\gamma, -S_\gamma, -S_\beta),
\]
we have
\[
H = X + \frac{S_\beta + S_\gamma}{(S_\beta + S_\gamma)(S_\beta + S_\gamma, -S_\gamma, -S_\beta)}(S_\beta + S_\gamma, -S_\gamma, -S_\beta).
\]
Therefore, its reflection in $BC$ is the point

$$X' = X - \frac{S_{\beta\gamma}}{(S_{\beta} + S_{\gamma})(S_{\beta\gamma} + S_{\gamma\alpha} + S_{\alpha\beta})} (S_{\beta} + S_{\gamma}, -S_{\gamma}, -S_{\beta}).$$

In homogeneous barycentric coordinates, this is

$$X' = (S_{\beta\gamma} + S_{\gamma\alpha} + S_{\alpha\beta})(0, S_{\gamma}, S_{\beta}) - S_{\beta\gamma}(S_{\beta} + S_{\gamma}, -S_{\gamma}, -S_{\beta})$$

$$= (-S_{\beta\gamma}(S_{\beta} + S_{\gamma}), S_{\gamma}(2S_{\beta\gamma} + S_{\gamma\alpha} + S_{\alpha\beta}), S_{\beta}(2S_{\beta\gamma} + S_{\gamma\alpha} + S_{\alpha\beta})).$$

The inferior of the Euler reflection is the point

$$X(125) = ((b^2 - c^2)(b^2 + c^2 - a^2) : (c^2 - a^2)^2(c^2 + a^2 - b^2) : (a^2 - b^2)^2(a^2 + b^2 - c^2)).$$

This is the intersection of the Euler lines of the triangles $AYZ$, $BZX$, $CXZ$, where $XYZ$ is the orthic triangle.

### 15.4 Circumcevian triangle

Let $P = (u : v : w)$. The lines $AP$, $BP$, $CP$ intersect the circumcircle again at $X$, $Y$, $Z$. The triangle $XYZ$ is called the circumcevian triangle of $P$. Since $X$ lies on the line $AP$, $X = (x : v : w)$ for some $x$. This point lies on the circumcircle if and only if

$$a^2vw + b^2xw + c^2xv = 0.$$

This gives $x = \frac{-a^2vw}{b^2w + c^2v}$. Therefore,

$$X = (-a^2vw : (b^2w + c^2)v : (b^2w + c^2)v).$$

Similarly,

$$Y = ((c^2u + a^2w)u : -b^2wu : (c^2u + a^2w)w), \quad Z = ((a^2v + b^2u)u : (a^2v + b^2u)v : -c^2uv).$$

**Proposition 15.3.** The circumcevian triangle of $P = (u : v : w)$ is perspective with the tangential triangle at

$$\left( a^2 \left( -\frac{a^4}{u^2} + \frac{b^4}{v^2} + \frac{c^4}{w^2} \right) : \cdots : \cdots \right).$$

**Proof.** The vertices of the tangential triangle are $(-a^2, b^2, c^2)$, $(a^2, -b^2, c^2)$, $(a^2, b^2, -c^2)$. The line joining $(-a^2, b^2, c^2)$ to $X$ is

$$\begin{vmatrix}
  x & y & z \\
  -a^2 & b^2 & c^2 \\
  -a^2vw & (b^2w + c^2)v & (b^2w + c^2)v \\
\end{vmatrix} = 0.$$
This is
\[(b^4w^2 - c^4v^2)x + a^2b^2w^2y - a^2c^2v^2z = 0.\]
Similarly, the lines joining \((a^2, -b^2, c^2)\) to \(Y\) and \((a^2, b^2, -c^2)\) to \(Z\) are
\[-a^2b^2w^2x + (c^4u^2 - a^4w^2)y + b^2c^2u^2z = 0,\]
\[a^2b^2v^2x - b^2c^2u^2y + (b^4v^2 - b^4u^2)z = 0.\]
These three lines concur at a point with coordinates given above.

**Example 15.2.**

1. \(G: X(22) = (a^2(-a^4 + b^4 + c^4) : \cdots : \cdots).\)

2. \(H: X(24) = \left(\frac{a^2(a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2)}{b^4 + c^4 - a^4} : \cdots : \cdots\right).\)

These two points are on the Euler line, and are the centers of similitude of the circum-circle and incircle of the tangential triangle.

## 15.5 The third Lemoine circle

Given a point \(P = (u : v : w)\), it is easy to find the equation of the circle \(\mathcal{C}_a\) through \(P, B, C\). Since \(\mathcal{P}(B) = \mathcal{P}(C) = 0\), the equation of the circle is

\[\mathcal{C}_a : a^2yz + b^2zx + c^2xy - (x + y + z) \cdot fx = 0\]
for some \(f\). Since the circle passes through \(P = (u : v : w)\), we must have

\[f = \frac{a^2vw + b^2wu + c^2uv}{u(u + v + w)}.\]

This circle \(\mathcal{C}_a\) intersects the lines \(AC\) and \(AB\) each again at another point. To find the intersection with \(AC\), we put \(y = 0\) in the equation of \((\mathcal{C}_a)\) and obtain \(b^2zx - fx(x + z) = 0, x((b^2 - f)z - fx)) = 0.\) Therefore, apart from \(C = (0, 0, 1)\), the circle \(\mathcal{C}_a\) intersects \(AC\) at

\[B_a = (b^2 - f : 0 : f) = (b^2u^2 + b^2uw - a^2vw - c^2uv : 0 : a^2vw + b^2wu + c^2uv).\]

Similarly, the circle \(\mathcal{C}_a\) intersects \(AB\) again at

\[C_a = (c^2 - f : f : 0) = (c^2u^2 + c^2wu - a^2vw - b^2wu : a^2vw + b^2wu + c^2uv : 0).\]

Similarly, with

\[g = \frac{a^2vw + b^2wu + c^2uv}{v(u + v + w)} \quad \text{and} \quad h = \frac{a^2vw + b^2wu + c^2uv}{w(u + v + w)},\]

the circles \(\mathcal{C}_b\) through \(P, C, A\) and \(\mathcal{C}_c\) through \(P, A, B\) intersect the sidelines again at

\[C_b = (g : c^2-g : 0), \quad A_b = (0 : a^2-g : g), \quad A_c = (0 : h : a^2-h), \quad B_c = (h : 0 : b^2-h).\]
Note the lengths of the segments:

\[ AB_a = \frac{f}{b^2} \cdot b = \frac{f}{b}, \quad AB_c = \frac{b^2 - h}{b^2} \cdot b = \frac{b^2 - h}{b}, \]

and

\[ AC_a = \frac{f}{c^2} \cdot c = \frac{f}{c}, \quad AC_b = \frac{c^2 - g}{c^2} \cdot c = \frac{c^2 - g}{c}. \]

The four points \( B_a, B_c, C_a, C_b \) are concyclic if and only if

\[ AB_a \cdot AB_c = AC_a \cdot AC_b \implies \frac{f(b^2 - h)}{b^2} = \frac{f(c^2 - g)}{c^2} \implies \frac{b^2}{c^2} = \frac{h}{g} = \frac{v^2}{w^2}. \]

Likewise, the four points \( C_b, C_a, A_b, A_c \) are concyclic if and only if \( \frac{c^2}{a^2} = \frac{w}{u} \), and the four points \( A_c, A_b, B_c, B_a \) are concyclic if and only if \( \frac{a^2}{b^2} = \frac{u}{v} \).

By the principle of 6 concyclic points, the six points \( A_b, A_c, B_c, B_a, C_a, C_b \) are concyclic if and only if

\[ u : v : w = a^2 : b^2 : c^2, \]

namely, \( P = (u : v : w) = (a^2 : b^2 : c^2) \), the symmedian point. The circle \( C \) containing these 6 points is the third Lemoine circle.

For this choice of \( P \),

\[ f = \frac{3b^2c^2}{a^2 + b^2 + c^2}, \quad g = \frac{3c^2a^2}{a^2 + b^2 + c^2}, \quad h = \frac{3a^2b^2}{a^2 + b^2 + c^2}. \]

With respect to the circle \( C \) containing these 6 points, we have

\[ \mathcal{P}(A) = \frac{f(b^2 - h)}{b^2} = \frac{3b^2c^2 \cdot b^2(b^2 + c^2 - 2a^2)}{b^2(a^2 + b^2 + c^2)^2} = \frac{3b^2c^2(b^2 + c^2 - 2a^2)}{(a^2 + b^2 + c^2)^2}. \]

Similarly,

\[ \mathcal{P}(B) = \frac{3c^2a^2(c^2 + a^2 - 2b^2)}{(a^2 + b^2 + c^2)^2}, \quad \mathcal{P}(C) = \frac{3a^2b^2(a^2 + b^2 - 2c^2)}{(a^2 + b^2 + c^2)^2}. \]

From these, we obtain the equation of the third Lemoine circle:

\[ (a^2 + b^2 + c^2)^2(a^2yz + b^2zx + c^2xy) - 3(x + y + z)(b^2c^2(b^2 + c^2 - 2a^2)x + c^2a^2(c^2 + a^2 - 2b^2)y + a^2b^2(a^2 + b^2 - 2c^2)z) = 0. \]
Chapter 16

The Brocard triangles and the Brocard circle

16.1 The first Brocard triangle

Consider the lines joining the vertices to the two Brocard points. Pairwise, they intersect at three points which form the Kiepert triangle \( \mathcal{K}(-\omega) \).

Since \( S_\omega = S_\alpha + S_\beta + S_\gamma \),

\[
B\Omega_\leftarrow \cap C\Omega_\rightarrow = X(-\omega) = (-a^2 : S_\gamma - S_\omega : S_\beta - S_\omega)
= (-a^2 : -S_\alpha - S_\beta : -S_\alpha - S_\gamma)
= (-a^2 : -c^2 : -b^2)
= (a^2 : c^2 : b^2).
\]

Similarly,

\[
C\Omega_\leftarrow \cap A\Omega_\rightarrow = Y(-\omega) = (c^2 : b^2 : a^2),
A\Omega_\leftarrow \cap B\Omega_\rightarrow = Z(-\omega) = (b^2 : a^2 : c^2).
\]
We call this the first Brocard triangle. It is perspective with $ABC$ at

$$K(-\omega) = \left( \frac{1}{S_\alpha - S_\omega} : \frac{1}{S_\beta - S_\omega} : \frac{1}{S_\gamma - S_\omega} \right) \sim \left( \frac{1}{a^2} : \frac{1}{b^2} : \frac{1}{c^2} \right) = K^*.$$ 

**Proposition 16.1.** The homogeneous barycentric coordinates of the Brocard points are

$$\Omega_\leftarrow = \left( \frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2} \right),$$

$$\Omega_\rightarrow = \left( \frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2} \right).$$

**Proof.** Since $\Omega_\leftarrow = BX(-\omega) \cap CY(-\omega)$, we compute its coordinates as follows.

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X(-\omega)$</td>
<td>$a^2$</td>
<td><strong><em>$</em>$</strong></td>
<td>$b^2$</td>
</tr>
<tr>
<td>$Y(-\omega)$</td>
<td>$c^2$</td>
<td>$b^2$</td>
<td><strong><em>$</em>$</strong></td>
</tr>
<tr>
<td>$\Omega_\leftarrow$</td>
<td>$c^2a^2$</td>
<td>$a^2b^2$</td>
<td>$b^2c^2$</td>
</tr>
</tbody>
</table>

Therefore, $\Omega_\leftarrow = \left( \frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2} \right)$; similarly for $\Omega_\rightarrow$. \qed

### 16.2 The Brocard circle

**Proposition 16.2.** (a) The circumcircle of $K(-\omega)$ contains the Brocard points $\Omega_\rightarrow$ and $\Omega_\leftarrow$.

(b) The first Brocard triangle $K(-\omega)$ is oppositely similar to $ABC$.

**Proof.** (a) We first show that $\Omega_\rightarrow, \Omega_\leftarrow, Y, Z$ are concyclic.

$$\angle(Y\Omega_\rightarrow, \Omega_\rightarrow Z) = \angle(A\Omega_\rightarrow, \Omega_\rightarrow B)$$

$$= \angle(A\Omega_\rightarrow, b) + \angle(b, c) + \angle(c, \Omega_\rightarrow B)$$

$$= \omega + \angle(b, c) + (-\omega)$$

$$= \angle(b, c)$$

$$= -\omega + \angle(b, c) + \omega$$

$$= \angle(C\Omega_\leftarrow, b) + \angle(b, c) + \angle(c, \Omega_\leftarrow A)$$

$$= \angle(C\Omega_\leftarrow, \Omega_\leftarrow A)$$

$$= \angle(Y\Omega_\leftarrow, \Omega_\leftarrow Z).$$

Similarly, $\Omega_\rightarrow, \Omega_\leftarrow, Z, X$ are concyclic, so are $\Omega_\rightarrow, \Omega_\leftarrow, X, Y$. This means that the five points $\Omega_\rightarrow, \Omega_\leftarrow, X, Y, Z$ are concyclic.

(b) It also follows that

$$\angle(YX, XZ) = \angle(Y\Omega_\rightarrow, \Omega_\rightarrow Z) = \angle(b, c).$$

For the same reason, we have $\angle(ZY, YX) = \angle(c, a)$ and $\angle(XZ, ZY) = \angle(a, b)$. This shows that $XYZ$ and $ABC$ are oppositely similar. \qed
Exercise

1. Show that for the first Brocard triangle \( XYZ \),
\[
\angle(\Omega \to X, X \Omega) = \angle(\Omega \to Y, Y \Omega) = \angle(\Omega \to Z, Z \Omega) = 2\omega.
\]

Proposition 16.3. The first Brocard triangle is inscribed in the circle with diameter \( OK \).

Proof. We prove this indirectly. Consider the circle \( C \) with diameter \( OK \). The perpendicular bisector of \( BC \) intersects \( C \) at \( O \) and a second point \( X' \). We claim that \( X' \) coincides with the vertex \( X = X(-\omega) \) of the first Brocard triangle.

Clearly, \( KX' \) is perpendicular to \( OX' \), and is parallel to \( BC \). This means that the distance from \( X' \) to \( BC \) is
\[
a^2 + b^2 + c^2 \cdot \frac{S}{a}.
\]
From this,
\[
cot \angle(a, BX') = \frac{a^2 + b^2 + c^2}{a^2 \cdot \frac{S}{a}} = \frac{a^2 + b^2 + c^2}{S} = \cot \omega.
\]
This means that \( \angle(a, BX') = \omega \), and \( X' \) lies on the line \( B\Omega \). Since \( X' \) lies on the perpendicular bisector of \( BC \), it is necessarily the vertex \( X(-\omega) \) of the first Brocard triangle.

The same reasoning shows that the other two vertices \( Y(-\omega) \) and \( Z(-\omega) \) also lie on the same circle with diameter \( OK \). \( \square \)

The circle with diameter \( OK \) is called the Brocard circle. It is also called the seven-point circle of triangle \( ABC \) since it contains, along with \( O, K \), the two Brocard points and the three vertices of \( K(-\omega) \).

Proposition 16.4. The two Brocard points \( \Omega \) and \( \Omega \) are symmetric with respect to the Brocard axis \( OK \).

Proof. It is enough to verify that the midpoint of \( \Omega \Omega \), namely, \( (a^2(b^2+c^2) : b^2(c^2+a^2) : c^2(a^2+b^2)) \), lies on the Brocard axis
\[
b^2c^2(b^2 - c^2)x + c^2a^2(c^2 - a^2)y + a^2b^2(a^2 - b^2)z = 0.
\]
\( \square \)

\[\text{\footnotesize{\(1\)}}\angle(\Omega \to X, X \Omega) = \angle(C \Omega \to, B \Omega) = \angle(C \Omega \to, a) + \angle(a, B \Omega) = \omega + \omega = 2\omega.\]
Proposition 16.5. The circumconic through the Brocard points is

\[(a^4 - b^2c^2)yz + (b^4 - c^2a^2)zx + (c^4 - a^2b^2)xy = 0.\]

Its center is the point

\[(((a^2 - bc)(a^2 + bc)(a^4 + a^2(b^2 + c^2) - (b^4 + b^2c^2 + c^4)) : \cdots : \cdots).\]

This circumonic also contains
(i) the Steiner point \(S_t\),
(ii) the Kiepert perspector \(K(\omega) = (\frac{1}{b^2 + c^2} : \frac{1}{c^2 + a^2} : \frac{1}{a^2 + b^2}) = X_{83}\),
(iii) \(X_{880} = (\frac{a^4 - b^2c^2}{a^2(b^2 - c^2)} : \cdots : \cdots).\)

Exercise

1. Let \(XYZ\) be the pedal triangle of \(\Omega_\rightarrow\) and \(X'Y'Z'\) be that of \(\Omega_\leftarrow\).

(a) Find the coordinates of these pedals.
(b) Show that \(Y'Z\) is parallel to \(BC\).
(c) The triangle bounded by the three lines \(Y'Z\), \(Z'X\) and \(X'Y\) is homothetic to triangle \(ABC\). What is the homothetic center? \(^2\)
(d) The triangles \(XYZ\) and \(Y'Z'X'\) are congruent.

\(^2\)The symmedian point.
The second Brocard triangle

Consider the circles $C_{AAB}$ through $A, B$, tangent to $CA$ at $A$, and $C_{CAA}$ through $A, C$, tangent to $AB$ at $A$.

Since $C_{AAB}$ passes through $A$ and $B$, its equation is of the form

$$a^2yz + b^2zx + c^2xy - (x + y + z) \cdot rz = 0$$

for some $r$. Since this is tangent to $CA$, if we set $y = 0$, the resulting quadratic equation $b^2zx - r(x + z)z = 0$ should have only one solution in $z = 0$. This means $r = b^2$. Therefore, we have

$$C_{AAB} : a^2yz + b^2zx + c^2xy - b^2(x + y + z)z = 0.$$ 

Similarly,

$$C_{CAA} : a^2yz + b^2zx + c^2xy - c^2(x + y + z)y = 0.$$ 

The radical axis of the two circles is the line $b^2z = c^2y$, which is clearly the $A$-symmedian. If we put $y = b^2$ and $z = c^2$ into the equation of $C_{AAB}$, we obtain $x = b^2 + c^2 - a^2$. Thus, The two circles intersect at

$$X_2 = (b^2 + c^2 - a^2 : b^2 : c^2).$$

The symmedian $AK$ intersects the circumcircle again at

$$A' = (-a^2 : 2b^2 : 2c^2).$$

The point $X_2$ is the midpoint of $AA'$. Therefore, $OX_2$ is perpendicular to the symmedian $AA'$, and $X_2$ lies on the Brocard circle.

The same reasoning applies to the two other pairs of circles, and we obtain the second Brocard triangle inscribed in the Brocard circle:

$$X_2 = (b^2 + c^2 - a^2 : b^2 : c^2),$$
$$Y_2 = (a^2 : c^2 + a^2 - b^2 : c^2),$$
$$Z_2 = (a^2 : b^2 : a^2 + b^2 - c^2).$$

Exercise

1. Show that the first and the second Brocard triangles are perspective and find the perspector. 

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3The centroid.
16.4 The Steiner and Tarry points

Proposition 16.6. Let \( XYZ \) be the first Brocard triangle.

(a) The parallels through \( A, B, C \) to the corresponding sides \( YZ, ZX, XY \) intersect at a point \( S_t \) on the circumcircle.

(b) The perpendiculars through \( A, B, C \) to the corresponding sides \( YZ, ZX, XY \) intersect at a point \( T_a \) the circumcircle.

(c) The points \( S_t \) and \( T_a \) are antipodal.

Proof. (a) From the coordinates of the vertices of the first Brocard triangle, the infinite point of the line \( YZ \) is \((b^2 - c^2, a^2 - b^2, c^2 - a^2)\). The parallel through \( A \) has equation \((c^2 - a^2)y - (a^2 - b^2)z = 0\). This line intersects the circumcircle at \( (\frac{1}{b^2-c^2}, \frac{1}{c^2-a^2}, \frac{1}{a^2-b^2}) \), which is the Steiner point \( S_t \). This is also true for the other two parallels.

(b) and (c) follow immediately from (a).

Remarks. (1) \( S_t \) is also the fourth intersection of the circumcircle with the Steiner circum-ellipse (with center \( G \)).

(2) \( T_a \) is also the fourth intersection of the circumcircle with the Kiepert hyperbola.
Exercise

1. $O$ and $K$ are the Tarry and Steiner points of the first Brocard triangle.
16.5  The third Brocard triangle

If the lines $OA$, $OB$, $OC$ intersect the Brocard circle again at $X_3$, $Y_3$, $Z_3$, we call $X_3Y_3Z_3$ the third Brocard triangle of $ABC$. These vertices have coordinates

$X_3 = (a^4 + (b^2 - c^2)^2 : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2)),$

$Y_3 = (a^2(b^2 + c^2 - a^2) : b^4 + (c^2 - a^2)^2 : c^2(a^2 + b^2 - c^2)),$

$Z_3 = (a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^4 + (a^2 - b^2)^2).$

**Proposition 16.7.** The second and third Brocard triangles are perspective at the circumcenter

$$(a^4S_A : b^4S_B : c^4S_C).$$

$$-b^2c^2(b^2 - c^2)x - c^2(c^4 + a^4 - c^2(a^2 + b^2))y + b^2(a^4 + b^4 - b^2(c^2 + a^2))x = 0.$$

**Remark.** The lines $X_1X_3$, $Y_1Y_3$, $Z_1Z_3$ bound a triangle perspective with $ABC$ at the orthocenter. The vertices are $(S_A : S_C : S_B)$ etc.

The points $X_{182}$, $X_{184}$ and the centroid are collinear.