Chapter 14

Hyperbolas

14.1 Hyperbolas

Hyperbola with two given foci

Given two points \( F \) and \( F' \) in a plane, the locus of point \( P \) for which the distances \( PF \) and \( PF' \) have a constant difference is a hyperbola with foci \( F \) and \( F' \).

Let \( F = (c, 0) \) and \( F' = (-c, 0) \). A point \((x, y)\) is on the hyperbola if and only if

\[
|\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2}| = 2a.
\]

Assume \( FF' = 2c \) and the constant difference \( |PF - PF'| = 2a \) for \( a < c \). Set up a coordinate system such that \( F = (c, 0) \) and \( F' = (-c, 0) \). A point \((x, y)\) is on the hyperbola if and only if

\[
|\sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2}| = 2a.
\]

This can be reorganized as

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{with} \quad b^2 := c^2 - a^2. \quad (14.1)
\]
The directrices and eccentricity of a hyperbola

If \( P = (x, y) \) is a point on the hyperbola \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \), its square distance from the focus \( F \) is 
\[ |PF| = \frac{c}{e} \left| x - \frac{a^2}{c} \right|. \]
Writing \( e := \frac{c}{a} \), we have \( PF = e \cdot \left| x - \frac{a^2}{c} \right| \). The hyperbola can also be regarded as the locus of point \( P \) whose distances from \( F \) (focus) and the line \( x = \frac{a}{e} \) (directrix) bear a constant ratio \( e > 1 \) (the eccentricity). Similarly, \( |PF'| = e \cdot \left| x + \frac{a^2}{c} \right| \). The point \( F' = (-ae, 0) \) and the line \( x = -\frac{a}{e} \) form another pair of focus and directrix of the same hyperbola.

There is an easy parametrization of the hyperbola: \( P(\theta) = (a \sec \theta, b \tan \theta) \).

Given a point \( T \) on the auxiliary circle \( \mathcal{C}(O, a) \), construct
(i) the tangent to the circle at \( T \) to intersect the \( x \)-axis at \( A \),
(ii) the line \( x = b \) to intersect the line \( OT \) at \( B \),
(iii) the “vertical” line at \( A \) and the “horizontal” line at \( B \) to intersect at \( P \).

The point \( P \) has coordinates \( (a \sec \theta, b \tan \theta) \), and is a point on the hyperbola \( \mathcal{H} \).

If \( Q \) is the pedal of \( T \) on the \( x \)-axis, then the line \( PQ \) is tangent to the hyperbola.

14.2 Tangents and normals of a hyperbola

Let \( P \) be a point on the hyperbola \( \mathcal{H} : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \). The circles with diameters \( PF_1 \) and \( PF_2 \) are tangent to the auxiliary circle \( \mathcal{C}(O, a) \). The line joining the points of tangency is tangent to \( \mathcal{H} \) at \( P \).

\[ 1 \cdot PF^2 = (x-c)^2 + y^2 = (x-c)^2 + b^2 \left( 1 - \frac{x^2}{a^2} \right) = \left( 1 - \frac{b^2}{a^2} \right) x^2 - 2cx + c^2 + b^2 = \frac{c^2}{a^2} x^2 - 2cx + a^2 = \left( \frac{c}{a} x - a \right)^2. \]
Exercise

1. If \( P \) and \( Q \) are on one branch of a hyperbola with foci \( F \) and \( G \), show that \( F \) and \( G \) are on one branch of a hyperbola with foci \( P \) and \( Q \).

2. Let \( P \) be a point on a hyperbola. The intersections of the tangent at \( P \) with the asymptotes, and the intersection of the normal at \( P \) with the axes lie on a circle through the four points passes through the center of the hyperbola.

The center of the circle is \( \left( \frac{a^2+b^2}{2a} \sec \theta, \frac{a^2+b^2}{2b} \tan \theta \right) \). The locus is the hyperbola

\[
4(a^2x^2 - b^2y^2) = (a^2 + b^2)^2.
\]
14.2.1 Evolute of a hyperbola

For the hyperbola \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) with parametrization \((x, y) = (a \sec t, b \tan t)\), the evolute is the parametric curve

\[
x = \frac{a^2 + b^2}{a} \cdot \sec^3 t, \quad y = -\frac{a^2 + b^2}{b} \cdot \tan^3 t.
\]
14.3 Rectangular hyperbolas

The line joining the points \( t, t' \) of the rectangular hyperbola \( xy = c^2 \) is

\[
x + tt'y - c(t + t') = 0.
\]

Example 14.1. Four points on the rectangular hyperbola \( xy = c^2 \) with parameters \( t_1, t_2, t_3, t_4 \) form an orthocentric system if and only if \( t_1t_2t_3t_4 = -1 \).

Proof. Consider three points on the hyperbola with parameters \( t_1, t_2, t_3 \). The perpendiculars from each one of them to the line joining the other two are the lines

\[
\begin{align*}
t_1t_2t_3x - t_1y + c(1 - t_1^2t_2t_3) & = 0, \\
t_1t_2t_3x - t_2y + c(1 - t_1t_2^2t_3) & = 0, \\
t_1t_2t_3x - t_3y + c(1 - t_1t_2^2t_3) & = 0.
\end{align*}
\]

If \( t_4 \) is such that \( t_1t_2t_3t_4 = -1 \), then it is easy to see that \( (x, y) = \left( ct_4, \frac{c}{t_4} \right) \) satisfies each of the above equations. Therefore, the orthocenter is the point with parameter \( t_4 \) on the hyperbola.

Example 14.2. Four points on the rectangular hyperbola \( xy = c^2 \) are with parameters \( t_1, t_2, t_3, t_4 \) are concyclic if and only if \( t_1t_2t_3t_4 = 1 \).

Proof. If the four points are on a circle with equation

\[
x^2 + y^2 + 2gx + 2fy + c_0 = 0
\]

then \( t_1, t_2, t_3, t_4 \) are the roots of the equation

\[
c^2t^2 + \frac{c^2}{t^2} + 2cgt + \frac{2cf}{t} + c_0 = 0,
\]

or

\[
c^2t^4 + 2cgt^3 + c_0t^2 + 2cft + c^2 = 0.
\]

From this we conclude that \( t_1t_2t_3t_4 = \frac{c^2}{c^2} = 1 \).

Example 14.3. \( PQ \) is a diameter of a rectangular hyperbola. The circle, center \( P \), passing through \( Q \), intersects the hyperbola at three points which form the vertices of an equilateral triangle.

Proof. Let \( P \) and \( Q \) be the points with parameters \( \tau \) and \( -\tau \). Since the points with parameters \( -\tau, t_1, t_2, t_3 \) are concyclic, \( -\tau \cdot t_1t_2t_3 = 1 \), and \( t_1t_2t_3 \cdot \tau = -1 \). This means that \( P \) is the orthocenter of the triangle with vertices \( t_1, t_2, t_3 \). This coincides with the circumcenter, and the triangle is equilateral.
Exercise

1. The pedal of the point \((p, q)\) on the line joining \((ct_1, \frac{c}{t_1})\) and \((ct_2, \frac{c}{t_2})\) is the point
\[
\left( \frac{t_2^2 t_1^2 p - t_1 t_2 q + (t_1 + t_2)c}{1 + t_1^2 t_2^2}, \frac{-t_1 t_2 p + q + t_1 t_2 (t_1 + t_2)c}{1 + t_1^2 t_2^2} \right).
\]

2. \(ABC\) is triangle with orthocenter \(H\), inscribed in a rectangular hyperbola with asymptotes \(L\) and \(L'\). Let \(X, Y, Z\) be the pedals of \(A, B, C\) on \(L\). Show that the perpendiculars from \(X\) to \(BC\), \(Y\) to \(CA\), and \(Z\) to \(AB\) are concurrent at the pedal of \(H\) on \(L'\).

14.4 A theorem on the tangents from a point to a conic

**Theorem 14.1.** Let \(P\) be the intersection of the tangents at \(T_1\) and \(T_2\) of an ellipse with foci \(F\) and \(F'\). The lines \(PF\) and \(PF'\) make equal angles with the tangents \(PT_1\) and \(PT_2\).

![Diagram of a conic with tangents and distance from foci](image)

**Proof.** (1) Let \(T_1 = (x_1, y_1)\). The tangents at \(T_1\) and \(T_2\) are the lines \(\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1\) and \(\frac{x x_2}{a^2} + \frac{y y_2}{b^2} = 1\). The distances from \(F = (c, 0)\) to these tangents are
\[
\begin{align*}
    d_1(F) &= \frac{1 - \frac{cx_1}{a^2}}{\sqrt{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}}} \\
    d_2(F) &= \frac{1 - \frac{cx_2}{a^2}}{\sqrt{\frac{x_2^2}{a^4} + \frac{y_2^2}{b^4}}}.
\end{align*}
\]
It follows that
\[
\frac{\sin T_1PF}{\sin FPT_2} = \frac{d_1(F')}{d_2(F)} = \frac{a^2 - cx_1}{a^2 - cx_2} \cdot \frac{\sqrt{x_1^2 + y_1^2}}{\sqrt{x_2^2 + y_2^2}}.
\]

Similarly, for \(F' = (-c, 0)\), we have
\[
\frac{\sin T_2PF'}{\sin F'PT_1} = \frac{d_2(F')}{d_1(F')} = \frac{a^2 + cx_2}{a^2 + cx_1} \cdot \frac{\sqrt{x_2^2 + y_2^2}}{\sqrt{x_1^2 + y_1^2}}.
\]

(2) It is routine to verify that
\[
\frac{\sin T_1PF}{\sin FPT_2} = \frac{\sin T_2PF'}{\sin F'PT_1}; \text{ equivalently,}
\frac{a^2 - cx_1}{a^2 - cx_2} \cdot \frac{a^2 + cx_1}{a^2 + cx_2} = \frac{x_1^2 + y_1^2}{x_2^2 + y_2^2}.
\]

This follows from
\[
\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} = \frac{x_1^2}{a^4} + \frac{1}{b^2} \left(1 - \frac{x_1^2}{a^2}\right) = \frac{a^4 - c^2x_1^2}{a^4b^2} = \frac{(a^2 - cx_1)(a^2 + cx_1)}{a^4b^2},
\]
and an analogous expression for \(\frac{x_2^2}{a^4} + \frac{y_2^2}{b^4}\).

(3) For convenience, let \(\angle T_1PT_2 = \Theta, \angle T_1PF = \theta\) and \(\angle T_2PF' = \phi\). We have
\[
\frac{\sin \theta}{\sin(\Theta - \theta)} = \frac{\sin \phi}{\sin(\Theta - \phi)},
\]
\[
\sin(\Theta - \theta) \sin \phi = \sin(\Theta - \phi) \sin \theta,
\]
\[
\cos(\Theta - \theta - \phi) - \cos(\Theta - \theta + \phi) = \cos(\Theta - \theta - \phi) - \cos(\Theta - \phi + \theta),
\]
\[
\cos(\Theta - \theta + \phi) = \cos(\Theta - \phi + \theta).
\]
Since \(\Theta \pm (\theta - \phi) < \pi\), we conclude that \(\Theta - \theta + \phi = \Theta - \phi + \theta\) and \(\theta = \phi\). \(\square\)

Remark. Similar calculations show that the theorem is also true for hyperbolas and parabola. For a parabola, the line joining \(P\) to the “second” focus is parallel to the axis.

Corollary 14.2. The foci of an inscribed conic of a triangle are isogonal conjugate points.

14.5 Miscellaneous exercises

1. (a) Find the equation of the normal at the point \((x_0, y_0)\) of the ellipse \(E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\).

(b) Show that if the line \(\frac{x}{p} + \frac{y}{q} = 1\) is normal to the ellipse \(E\), then \(a^2p^2 + b^2q^2 = (a^2 - b^2)^2\).

2. An ellipse with semi-axes \(a\) and \(b\) slides on the positive \(x\)- and \(y\)-axes. Find the locus of its center.
3. (Director circle) The tangents to the ellipse \( E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) with slope \( m \) are

\[ y = mx \pm \sqrt{m^2a^2 + b^2}. \]

Deduce that the locus of points from which the two tangents to \( E \) are perpendicular is the circle

\[ x^2 + y^2 = a^2 + b^2. \]

4. (Pair of tangents) (a) Show that the line \( y = mx + c \) touches the ellipse \( E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) if (and only if) \( c^2 = m^2a^2 + b^2 \).

(b) Deduce that if \( (p, q) \) is outside the ellipse, and \( p^2 \neq a^2 \), then the two tangents from \( (p, q) \) to the ellipse are given by

\[ (qx - py)^2 = b^2(x - p)^2 + a^2(y - q)^2. \]

5. (Chords of ellipse subtending a right angle at the center)

The line \( \frac{x}{p} + \frac{y}{q} = 1 \) intersects the ellipse \( E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) at two points \( P \) and \( Q \) such that \( \angle P O Q = \frac{\pi}{2} \) if and only if

\[ \frac{1}{p^2} + \frac{1}{q^2} = \frac{1}{a^2} + \frac{1}{b^2}. \]

Construct the circle, center \( O \), tangent to a line joining two vertices of the ellipse (not on the same axis). A chord of the ellipse cut out by a tangent of this circle subtends a right angle at the center.

6. Let \( B \) and \( B' \) be the endpoints of the minor axis of an ellipse. \( P \) and \( Q \) are points on the ellipse such that \( BP \) and \( BQ \) are perpendicular. Show that the lines \( BP \) and \( B'Q \) intersect on a fixed line.

7. Given an ellipse of eccentricity \( e > \frac{1}{\sqrt{2}} \), construct two circles with centers on the major axis, each passing through the center and tangent to the ellipse at two points.

8. Given an ellipse, construct an isosceles triangle with apex at a vertex on the minor axis and incircle concentric with the ellipse.
9. In a right triangle of sides \( a \) and \( b \), an ellipse of semi-axes \( h \), \( k \) (parallel to the sides) is inscribed. Show that

\[ 2(a - h)(b - k) = ab. \]

10. An ellipse is tangent to the four sides of a rectangle at their midpoints. Construct the circle tangent to the ellipse and two adjacent sides of the rectangle.

11. Find the locus of the vertices of a rectangular hyperbola with center \( O \) and tangent to the line \( x = a \). Hint: Use polar coordinates.

12. If \( P \) and \( Q \) are on one branch of a hyperbola with foci \( F \) and \( G \), show that \( F \) and \( G \) are on one branch of a hyperbola with foci \( P \) and \( Q \).